Solution, Exam in MMA 100 Topology, 7.5 HEC. 2012-June 4

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1. **[a]** Claim: The connected components are $A_n := [n, n+1)$. We observe first that $[n, m)$ is both open can closed. Indeed its complement is $A_n^c = (-\infty, n) \cup [m, \infty)$, $(-\infty, n) = \cup_{j=1}^{\infty}[n-j, n)$, and $[m, \infty) = \cup_{j=1}^{\infty}[m, m+j)$, which is open. Now A_n is open and the only nonempty open set in A_n is A_n itself. Thus A_n is not a union of two disjoint open sets, and A_n is connected. Any connected set B containing A_n, $A_n \subsetneq B$, can be written as $B = A_n \cup C$, with $C = B \cap A_n^c$ is open in B since A_n^c is open. Thus B is not connected and A_n is maximal connected subset.

   [b] Let $f(x) = [x] = \inf\{n; n \geq x\}$, the least integer not less than x. Then f is a map since any open set in $\mathbb{Z}$ is of a union of the one-point sets $\{n\}$, whose inverse image is $[n, n+1)$ which is open. Thus the inverse image of an open set in $\mathbb{Z}$ is open, completing the proof.

2. The one-point set $\{y\}$ is closed since Y is Hausdorff. The inverse image $f^{-1}(\{y\})$ is therefore a closed set in X since f is map. Now X is compact and any closed set in X is compact, and in particular $f^{-1}(\{y\})$ is compact.

3. Suppose f is map. Recall first that for any subset C, we have $C^0 \subset C$ is open and for any open set $D \subseteq C$ we have $C \subseteq C^0$. Now for any $A \subseteq Y$ we have $A^0 \subseteq A$ is open. Thus $f^{-1}(A^0)$ is open. But $f^{-1}(A^0) \subseteq f^{-1}(A)$, and $\left(f^{-1}(A)\right)^0$ is the largest subset contained in $f^{-1}(A)$. Thus $f^{-1}(A^0) \subseteq \left(f^{-1}(A)\right)^0$.

   Conversely suppose $f^{-1}(A^0) \subseteq \left(f^{-1}(A)\right)^0$ for any A. Let B be any open set in Y. We have $B^0 = B$. Thus $f^{-1}(B) = f^{-1}(B^0) \subseteq \left(f^{-1}(B)\right)^0 \subset f^{-1}(B)$. That is $f^{-1}(B) = \left(f^{-1}(B)\right)^0$ is open. f is a map by definition.

4. See the Textbook

5. **[a]** The identification map f is not open. Take A the open interval $A = (-\frac{1}{4}, \frac{1}{4})$ in $\mathbb{R}$ and let $B = f(A)$ be its image. The inverse image of B is $A \cup \mathbb{Z}$, which is not open, and thus B is not open. **[b]** $\mathbb{R}/\mathbb{Z}$ is not compact. Let $A_n = (n - \frac{1}{n+1}, n + \frac{1}{n+1}) \cup \cup_{|j| \geq n+1}^{\infty}(j - \frac{1}{n+2}, j + \frac{1}{n+2})$, and $B_n = f(A_n) = A_n/\mathbb{Z}$, $n \geq 2$. Then B_n is open in $\mathbb{R}/\mathbb{Z}$ since its inverse image is A_n which is open, for any n. $\{B_n\}$ is then a cover of $\mathbb{R}/\mathbb{Z}$ and has no finite subcover. Indeed any finitely many subsets in B_n is covered by $\{B_n, |n| \leq N\}$ for some big N. But the bigger family does not cover the point $N + \frac{1}{2}$.

6. **[a]** See the Textbook

   [b] Let $f : S \to S^2 \setminus \{n, s\}$ be the inclusion map. Let P be the orthogonal projection of $\mathbb{R}^3$ onto the x_1x_2-plane $\mathbb{R}^2$, and let $R : \mathbb{R}^2 - \setminus\{0\} \to S$ the projection onto S along the rays, i.e. $Rx = x/|x|$. Define $g : S^2 \setminus \{n, s\} \to S$ by $g = RP$. We have the maps $S \xrightarrow{f} S^2 \setminus \{n, s\} \xrightarrow{g} S \xrightarrow{f} S^2 \setminus \{n, s\}$. Now $g \circ f = Id$ and we claim $f \circ g \sim_{\ homotopy} Id$. For any $x \in S^2 \setminus \{n, s\}$, denote $\theta(x)$ the angle between x and the x_1x_2-plane and $\mathbb{P}_x$ the plane passing the three points, 0, $g(x) \in \mathbb{R}^2$ and x. We let $F_t = F(\cdot, t) : S^2 \setminus \{n, s\} \to S^2 \setminus \{n, s\}$ be the map defined by

   $$F_t(x) \text{ is the point in the plane } \mathbb{P}_x \text{ of angle } t\theta(x) \text{ to the vector } g(x),$$

   for $0 \leq t \leq 1$. Thus $F_1 = I$ and $F_0 = g = f \circ g$. Thus completes the proof of the claim.

7. Let $f : S \times \mathbb{R} \to S \times \mathbb{R}$, $(c, x) \mapsto (e^{-ix}c, x)$. Then f is a homeomorphism and its inverse is given by

$$f^{-1} : (c, x) \mapsto (e^{ix}c, x).$$

By conjugating f the action $n : (c, x) \mapsto n \cdot (c, x)$ $\mathbb{Z}$ on $S \times \mathbb{R}$ defines an action $f^{-1} \circ n \circ f$: $(d, x) \mapsto f \circ n \circ f^{-1}$ on $S \times \mathbb{R}$. To find the quotient space $S \times \mathbb{R}$ under the orginal action is the same as to find that under $f \circ n \circ f^{-1}$. But now

$$f^{-1} \circ n \circ f : (d, x) \mapsto (e^{-ix}d, x) \mapsto (e^{-ix}de^{in}, x+n) \mapsto (e^{-ix}de^{in}e^{-i(x+n)}, x+n) = (d, x+n).$$

In otherwords the action $f^{-1} \circ n \circ f$ is the action of $\mathbb{Z}$ on the cylinder $S \times \mathbb{R}$ translating $\mathbb{R}$ with integers. Thus the orbit space $S \times \mathbb{R}/\mathbb{Z}$ (under the action $f^{-1} \circ n \circ f$) is the torus $S \times S$.

8. We prove that Y is homeomorphic to the punctured plane $\mathbb{R}^2 \setminus \{0\}$. The space Y consists of the lines $[x]$ with $x = (x_1, x_2, x_3)$, and $x_3 \neq 0$ and $(x_1, x_2) \neq (0, 0)$. By multiplying the representative x by $\frac{1}{x_3}$ we see that each line $[x]$ in Y can be uniquely written as $[y]$ with $y = (y_1, y_2, 1)$, and $(y_1, y_2) \neq (0, 0)$, in other words $(y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$. Now consider the map $f : Y \to \mathbb{R}^2 \setminus \{0\}$, $[x] \mapsto \frac{1}{x_3}(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ and $g : \mathbb{R}^2 \setminus \{0\} \to Y$, $(y_1, y_2) \mapsto [(y_1, y_2, 1)]$. We have then $f \circ g = Id$, $g \circ f = Id$. Thus Y is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

[a] Y is path-connected since $\mathbb{R}^2 \setminus \{0\}$ is.

[b] The fundamental group of Y is that of $\mathbb{R}^2 \setminus \{0\}$, which is $\mathbb{Z}$.