Mathematical Sciences (MV), GU
June 4, 2012, 8:30-12:30.

No aids (closed book, closed notes).
Phone: Urban Larsson, 0703-088304
Presence of teacher: ~ 9.30 and ~ 11.30

Exam in MMA 100 Topology, 7.5 HEC.

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1. We define a topology $\mathcal{U}$ on the real line $\mathbb{R}$ so that the collection of intervals $\{[n, m); n, m \in \mathbb{Z}\}$ forms a base of $\mathcal{U}$, i.e., $\mathcal{U}$ consists of unions and finite intersections of these intervals $[n, m)$.

   [a] Find the connected components of $(\mathbb{R}, \mathcal{U})$.

   [b] Prove that the identity map $I_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ can be extended to a map from $(\mathbb{R}, \mathcal{U})$ to $\mathbb{Z}$. Here $\mathbb{Z}$ is equipped with the discrete topology. (1+2p)

2. Let $f : X \to Y$ be a map where X is compact and Y is Haussdorff. Prove that for any $y \in Y$ the inverse image $f^{-1}(\{y\})$ is compact. (3p)

3. Prove that a function $f : X \to Y$ is a map if and only if $f^{-1}(A^0) \subseteq \left(f^{-1}(A)\right)^0$ for any subset $A \subseteq Y$. (Recall that B^0 denotes the set of interior points of a subset B.) (3p)

4. Let G be a topological group and let K be the connected component containing the identity of G. Prove that K is a closed normal subgroup of G. (3p)

5. Consider the identification space $\mathbb{R}/\mathbb{Z}$ by identifying all the integers $\mathbb{Z}$ as a single point and each non-integer point as itself. Prove or disprove the following claims: **[a]** The natural identification map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is open (i.e. taking open sets to open sets); **[b]** $\mathbb{R}/\mathbb{Z}$ is compact. (1+2p)

6. **[a]** Formulate the definition that two spaces A and B are of the same homotopy type.

   [b] Let n and s be the north and south poles of the 2-sphere S^2. Prove that $X = S^2 \setminus \{n, s\}$ (i.e. the sphere with the south and north poles removed) is of the same homotopy type as the circle S. (1+2p)

7. We parametrize the circle S as the subset $S = \{c \in \mathbb{C}; |c| = 1\}$ in the plane $\mathbb{C}$ and consider the action of $\mathbb{Z}$ on $S \times \mathbb{R}$ defined by $n \cdot (c, x) = (ce^{in}, x + n)$, $n \in \mathbb{Z}$. Prove that the orbit space $S \times \mathbb{R}/\mathbb{Z}$ is homeomorphic to the torus $S \times S$. (Hint: Consider the homeomorphism $(c, x) \to (e^{-ix}c, x)$ of $S \times \mathbb{R}$ and the induced map on $S \times \mathbb{R}/\mathbb{Z}$.) (3p)

8. Recall that the projective space $\mathbb{P}^2$ is the orbit space $\mathbb{R}^3 \setminus \{0\}/\mathbb{R}^*$ where $\mathbb{R}^*$ acts on $\mathbb{R}^3 \setminus \{0\}$ by scalar multiplication (i.e. $\mathbb{P}^2$ is the set of lines $[x]$ through 0 in $\mathbb{R}^3$). Let X be the subset of $\mathbb{P}^2$ of lines in x_1x_2-plane, $X = \{[x] \in \mathbb{P}^2; x = (x_1, x_2, 0)\}$ and $[e_3]$ the x_3-axis, and let Y be $\mathbb{P}^2$ with X and $[e_3]$ removed, $Y = \mathbb{P}^2 \setminus (X \cup \{[e_3]\})$.

   [a] Prove that Y is path-connected.

   [b] Find the fundamental group of Y. (1+2p)

Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good). GZ