Mathematical Sciences (MV), GU
March 9, 2012, 8:30-12:30.

No aids (closed book, closed notes).
Phone: Ragnar Freij, 0703-088304
Presence of teacher: ~ 9.30 and ~ 11.30

Exam in MMA 100 Topology, 7.5 HEC.

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1. Let $\mathcal{U}$ be the collection of the subsets of $\mathbb{R}$ of the form (a, ∞), $a \in \mathbb{R}$ together with $\mathbb{R}$ and the empty set $\emptyset$. The collection $\mathcal{U}$ defines then a topology on $\mathbb{R}$. Prove that a subset A in $(\mathbb{R}, \mathcal{U})$ is compact if and only if $\inf A$ exists and is in A, where $\inf A$ is the infimum of the set A. (3p)

2. Suppose A and B are two compact subset of a Haussdorff space X and $A \cap B = \emptyset$. Prove that there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. (3p)

3. Let $M_1(I, S^1)$ be the space of all loops in the unit circle S^1 based at $1 \in S^1$ equipped with the metric $d(f, g) = \sup_{x \in I} |f(x) - g(x)|$, and let e_1 be the trivial loop at 1.

   [a] Prove that if $f \in M_1(I, S^1)$ and $d(f, e_1) < 2$ then f is homotopic to e_1.

   [b] Prove that if $f, g \in M_1(I, S^1)$ and $d(f, g) < 2$ then f is homotopic to g. (Hint: Use [**a**])

   [c] Prove that $M_1(I, S^1)$ has infinitely many connected components. (Hint: Use [**b**])

   (1p+1p+1p)

4. Formulate the definition of a covering map $\pi : \tilde{X} \to X$. Prove by definition that if $\pi : \tilde{X} \to X$ is a covering then π is an open map. (3p)

5. Prove that a connected open subset of the Euclidean space $\mathbb{R}^n$ is path-connected. (3p)

6. Let A be the space $\mathbb{R}^3$ with the z-axis removed. Find the fundamental group $\pi_1(A)$. Give detailed proof of your claim. (3p)

7. Suppose H is a closed subgroup of a topological group G (with identity e).

   [a] Suppose $g_0 \notin H$. Prove that there exist a neighborhood U_0 of g_0 and neighborhoods U, V of e such that $U_0 \cap H = \emptyset$, $U^{-1} g_0 V \subseteq U_0$, where $U^{-1} g_0 V$ is the short-hand notation for the set $\{u^{-1} g_0 v; u \in U, v \in V\}$.

   [b] Prove that G/H is Haussdorff. (Hint: Enough to consider the two point $[g_0] = g_0 H$ and $[e]$ in **[a])**

   (1.5p+1.5p)

8. View $\mathbb{R}^{n+1}$ as a subspace $\mathbb{R}^{n+k+1}$ by identifying $x \in \mathbb{R}^{n+1}$ with $(x, 0) \in \mathbb{R}^{n+k+1}$, and thus the sphere S^n as a subset of S^{n+k}.

   [a] Prove that $S^3 \setminus S^2$ is disconnected.

   [b] Prove that $S^3 \setminus S^1$ is path-connected and find its fundamental group. (Hint: Use the identification S^3 as the orbit space $(\mathbb{R}^4 \setminus \{0\})/\mathbb{R}^+$ or the stereographic projection of $S^3 \setminus \{N\}$ onto $\mathbb{R}^3$ to visualize $S^3 \setminus S^1$.)

   (1p+2p)

Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good). GZ