Mathematical Sciences (MV), GU
Thursday, August 21, 2014, 8:30-12:30.

No aids (closed book, closed notes).
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Presence of teacher: $\sim 9:30$ and ~ 11.30

Exam in MMA 100 Topology, 7.5 HEC.

1. Let X be a topological space. Prove or disprove the following statements for subsets of X:

   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ($\overline{A}$ stands for the closure of A).

   (b) $\partial A \cap \partial B = \partial(A \cap B)$. ($\partial A = \overline{A} \cap \overline{A^c}$ stands for the boundary of A).

   Solution. (a) True statement. We have $A \subset A \cup B$, $B \subset A \cup B$, and thus $\bar{A} \subset \overline{A \cup B}$, $\bar{B} \subset \overline{A \cup B}$, implying $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Conversely if $x \notin \bar{A} \cup \bar{B}$ then $x \notin \bar{A}$ and $x \notin \bar{B}$. There exist U_1 and U_2 such that $U_1 \cap A = \emptyset$, $U_2 \cap B = \emptyset$. Take $U = U_1 \cap U_2$, a neighborhood of x. Then

   $$U \cap (A \cup B) = (U \cap A) \cup (U \cap B) \subset (U_1 \cap A) \cup (U_2 \cap B) = \emptyset.$$

   Thus $x \notin \overline{A \cup B}$. This finished the proof.

   (b) False. Take $A = (-1, 0)$, $B = (0, 1)$.

2. Let $\mathbb{R}^* = \{x \in \mathbb{R}; x \neq 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$ be the set of non-zero respectively positive real numbers. Consider the map $f : \mathbb{R}^* \to \mathbb{R}^+, f(x) = x^2$. We define a new topology $\mathcal{N}$ on $\mathbb{R}^*$ be requiring that the open sets are of the form $f^{-1}(O)$, where $O \subset \mathbb{R}^+$ are open sets in $\mathbb{R}^+$ (equpped with the standard Euclidean topology). Prove that $(\mathbb{R}^*, \mathcal{N})$ is connected and non-Haussdorff.

   Solution. Let $\mathbb{R}^* = A \cup B$ be a disjoint union of two non-empty open sets. We shall prove this leads to a contradiction. By the definition of open sets in $\mathbb{R}^*$ and that any open set in $\mathbb{R}^+$ is a disjoint union of open intervals (a, b), $0 < a < b$, we see that A is a disjoint union of open intervals of the form (a, b), $(-b, -a)$. In otherwords f is an open map, and we have $R^+ = f(A) \cup f(B)$ and $f(A)$ and $f(B)$ are open sets. Now if $f(a) = f(b)$ for $a \in A, b \in B$ we have that $a = \pm b$ but A is open and contains $\pm a$ thus $b = \pm a$ is in A, contradicting $A \cap B = \emptyset$. In other words, we have $R^+ = f(A) \cup f(B)$, a union of two disjoint open sets, contradicting that $\mathbb{R}^+$ is connected.

   Take the points $1, -1$. They are distincts but are always in one neighborhood. Thus it is not Haussdorff.

3. Let S be the unit circle in the plane $\mathbb{R}^2$, and X be the subset of the product $S \times S$ consisting of non-parallell unit vectors (x, y). We equipp $S \times S$ with the product topology and X the subset topology. Answer the following questions with proofs: (a) Is X a compact set? (b) Is the map $f : X \to S$, $f(u, v) = u$, a closed map (i.e., mapping closed sets to closed sets)?

   Solution. (a) X is not compact. $S \times S$ is a compact set and Haussdorff. Thus $X \subset S \times S$ is compact iff it is closed. But the complement of X is the set $\{(x, \pm x) \in S \times S, x \in S\}$, and is closed (in fact compact). Thus X is open in $S \times S$ and is not closed (as the only open and closed non-empty set in $S \times S$ is the total space since it is connected.) (Alternatively we may take a sequence of the form (e_1, p_n) in X with p_n approaching 1 and see that X is not closed.)

   (b) Not a closed map. Let e_1 be the unit basis vector in S and consider $U = \{(p, e_1) \in X; p \neq e_1\}$ then U is a closed set since its closure in X is itself. However the projection p maps U onto the set $\{p; p \neq e_1\}$ in S which is clearly not closed.

4. Prove that the orthogonal group $O(2n+1)$ and $\mathbb{Z}_2 \times SO(2n+1)$ are homeomorphic as topological spaces and isomorphic as groups. Prove that $O(2)$ and $\mathbb{Z}_2 \times SO(2)$ are not isomorphic.

   Solution. We realize $\mathbb{Z}_2$ as $\{\pm 1\}$. Notive first each element $x \in O(m)$ has its determinant being ± 1. Now we consider the map $f : O(2n+1) \to \mathbb{Z}_2 \times SO(2n+1), x \mapsto (\det x, (\det x)x)$. Then $\det((\det x)x) = (\det x)^{2n+1} \det x = 1$, so it is well-defined, and is a homormorphism. It is also onto since any element $(\epsilon, y) \in \mathbb{Z}_2 \times SO(2n+1)$ can be written as $(\det x, (\det x)x)$ for $x = \epsilon y$. Thus they are isomorphic as groups. Now both spaces are compact we have f is also homeomorphism.

   However $O(2)$ and $\mathbb{Z}_2 \times SO(2)$ are not isomorphic, since the former is non-commutative, e.g. a reflection in y-axis and a rotation $\frac{\pi}{2}$ are not commuting, and the latter is commutative.

5. Let X be a topological space. Prove the following: (a) If C is a convex subset of $\mathbb{R}^n$ then any two maps $f, g : X \to C$ are homotopic to each other. (b) If $f, g : X \to S^2$ are maps to the unit sphere such that $f(x) + g(x) \neq 0$ then they are homotopic to each other.

   Solution. (a) We choose $h(x,t) = (1-t)f(x) + tg(x)$. Then $h : X \times I \to X$ is well-defined since C is convex, and is a homotopy between f and g.

   (b) We can take
$$h(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}, (x,t) \in X \times I$$
   $h(x,t)$ is well-defined if $(1-t)f(x) + tg(x) \neq 0$. However if $(1-t)f(x) + tg(x) = 0$ then $(1-t)f(x) = -tg(x)$, and further by taking norm, $1 - t = t$, i.e, $t = \frac{1}{2}$, which in turns gives $\frac{1}{2}f(x) + \frac{1}{2}g(x) = 0$, and $f(x) + g(x) = 0$ and is excluded by our assumption. Now h defines a homotopy between f and g.

6. Let S^1 be the unit circle in the plane written in complex coordinates as $S^1 = \{e^{i\theta}; 0 \leq \theta < 2\pi\}$. Consider the action of $\mathbb{Z}$ on $X := \mathbb{R} \times S^1$, $n : (x,u) \mapsto (x+n, e^{in\frac{\pi}{2}}u)$. Find the homotopy group $\pi_1(X/\mathbb{Z})$ of the orbit space $X/\mathbb{Z}$.

   Solution. Consider the map
$$f : X \to X, (x,u) \mapsto (x, e^{i\frac{\pi}{2}x}u)$$
   Then f is an homeomorphim of X with the inverse given by
$$f^{-1} : X \to X, (x,v) \mapsto (x, e^{-i\frac{\pi}{2}x}v)$$
   Denote the action by α. The conjugation $n \to f^{-1} \circ \alpha(n) \circ f$ defines an action of $\mathbb{Z}$ on X. The corresponding orbit space is homeomorphic to the orbit space $X/(\mathbb{Z}, \alpha)$, which is given by the map $[x] \to [f^{-1}x]$, sending the orbits of α to orbits of $f^{-1} \circ \alpha(n) \circ f$. However $f^{-1} \circ \alpha(n) \circ f$ is
$$\begin{aligned}(x,u) &\to (x, e^{i\frac{\pi}{2}x}u) \to (x+n, e^{in\frac{\pi}{2}}e^{i\frac{\pi}{2}x}u) = (x+n, e^{i(x+n)\frac{\pi}{2}}u) \\ &\to (x+n, e^{-i\frac{\pi}{2}(x+n)}e^{i(x+n)\frac{\pi}{2}}u) = (x+n, u),\end{aligned}$$
   and is acting on x along. Thus the corresponding orbit space $X/(\mathbb{Z}, f^{-1} \circ \alpha(n) \circ f)$ is $(\mathbb{R}/\mathbb{Z}) \times S^1 = S^1 \times S^1$. The homotopy group is thus $\mathbb{Z}^2$.

7. Formulate and prove the Brower's Fixed Point Theorem (for a disk in the plane).

   See the textbook.

8. Let $\mathbb{P}^n$ be the projective space of lines $[x] = \mathbb{R}x$ in $\mathbb{R}^{n+1}$. We consider the map $f : \mathbb{P}^1 \to \mathbb{P}^2, x = [x_1, x_2] \mapsto [x_1, x_2, x_1 + x_2]$. Prove that f is well-defined and find the homotopy group of $\mathbb{P}^2 \setminus f(\mathbb{P}^1)$.

Solution. f is linear and thus maps line to line or to the zero vector. However $f(x) \neq 0$ if $x \neq 0$. Thus f well-defined on the projective space.

The image $f(\mathbb{P}^1)$ consists of all lines in the plan $x_3 = x_1 + x_2$. Thus its complement $\mathbb{P}^2 \setminus f(\mathbb{P}^1)$ consists of all lines $[x] = [(x_1, x_2, x_3)]$ with $x_3 \neq x_1 + x_2$. We prove first that this is a path-connected set. Each point in the projective space $\mathbb{P}^2$ can be viewed as a pair of antipodal points on the sphere S^2. Thus $\mathbb{P}^2 \setminus f(\mathbb{P}^1)$ consists of pairs of antipodal points $\pm x$ on the sphere such that $x_3 \neq x_1 + x_2$. The plane $x_3 = x_1 + x_2$ cuts the sphere in two path connected pieces, an upper sphere with $x_3 > x_1 + x_2$, and lower sphere $x_3 < x_1 + x_2$,. However each point $x_3 > x_1 + x_2$ has its antipodal points $(y_1, y_2, y_3) = -x$ satisfying $y_3 < y_1 + y_2$, and the two points represent the same point on the projective space. Thus $\mathbb{P}^2 \setminus f(\mathbb{P}^1)$ is path-connected, as any two points can be connected by a path in the upper half-sphere considered as path in the projective space. (This can also be obtained from the following arguments using the homotopy.)

We claim that the homotopy group $\mathbb{P}^2 \setminus f(\mathbb{P})$ is trivial. (Indeed it is just the upper-half sphere, intuitively.) We consider the map

$$g : \mathbb{P}^2 \setminus f(\mathbb{P}^1) \to \mathbb{R}^+, \quad [x_1, x_2, x_3] \mapsto |x_3 - x_1 - x_2|$$

and

$$j : \mathbb{R}^+ \to \mathbb{P}^2 \setminus f(\mathbb{P}^1), \quad s \mapsto [0, 0, s]$$

By the definition we see that g, j is well-defined and continuous. Clearly $g \circ j = Id$, the identity map. On the other hand,

$$j \circ g : [x_1, x_2, x_3] \mapsto [0, 0, |x_3 - x_1 - x_2|]$$

To prove this is homotopic to the identify we take

$$h(t, x) = [(1-t)x_1, (1-t)x_2, x_3 - tx_1 - tx_2], x \in \mathbb{P}^2 \setminus f(\mathbb{P}^1), 0 \leq t \leq 1$$

Now to check h is well-defined we observe that if the vector $((1-t)x_1, (1-t)x_2, x_3 - tx_1 - tx_2) = 0$, then $x_3 = x_1 + x_2$ and x is zero or is on the plane so defined, which is in the image of f. Furthermore $h(0, x) = Id$ and $h(1, x) = [0, 0, x_3 - x_1 - x_2] = [0, 0, |x_3 - x_1 - x_2|] = j \circ g$, proving the claim and thus the homotopy group is the same as $\mathbb{R}^+$, which is the trivial group 0.