Mathematical Sciences (MV), GU
Thursday, August 21, 2014, 8:30-12:30.

No aids (closed book, closed notes).
Phone: 0703-088304 Christoffer Standar
Presence of teacher: $\sim 9:30$ and $\sim 11.30$

# Exam in MMA 100 Topology, 7.5 HEC.

1. Let $X$ be a topological space. Prove or disprove the following statements for subsets of $X$:

   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ($\overline{A}$ stands for the closure of $A$).

   (b) $\partial A \cap \partial B = \partial(A \cap B)$. ($\partial A = \overline{A} \cap \overline{A^c}$ stands for the boundary of $A$).

2. Let $\mathbb{R}^* = \{x \in \mathbb{R}; x \neq 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$ be the set of non-zero respectively positive real numbers. Consider the map $f : \mathbb{R}^* \to \mathbb{R}^+, f(x) = x^2$. We define a new topology $\mathcal{N}$ on $\mathbb{R}^*$ be requiring that the open sets are of the form $f^{-1}(O)$, where $O \subset \mathbb{R}^+$ are open sets in $\mathbb{R}^+$ (equpped with the standard Euclidean topology). Prove that $(\mathbb{R}^*, \mathcal{N})$ is connected and non-Haussdorff.

3. Let $S$ be the unit circle in the plane $\mathbb{R}^2$ and $X$ be the subset of the product $S \times S$ of consisting of non-parallell unit vectors $(x, y)$. We equipp $S \times S$ with the product topology and $X$ the subset topology. Answer the following questions with proofs: (a) Is $X$ a compact set? (b) Is the map $f : X \to S$, $f(u, v) = u$, a closed map (i.e., mapping closed sets to closed sets)?

4. Prove that the orthogonal group $O(2n + 1)$ and $\mathbb{Z}_2 \times SO(2n + 1)$ are homeomorphic as topological spaces and isomorphic as groups. Prove that $O(2)$ and $\mathbb{Z}_2 \times SO(2)$ are not isomorphic.

5. Let $X$ be a topological space. Prove the following: (a) If $C$ is a convex subset of $\mathbb{R}^n$ then any two maps $f, g : X \to C$ are homotopic to each other. (b) If $f, g : X \to S^2$ are maps to the unit sphere such that $f(x) + g(x) \neq 0$ then they are homotopic to each other.

6. Let $S^1$ be the unit circle in the plane written in complex coordinates as $S^1 = \{e^{i\theta}; 0 \leq \theta < 2\pi\}$. Consider the action of $\mathbb{Z}$ on $X := \mathbb{R} \times S^1$, $n : (x, u) \mapsto (x + n, e^{in\frac{\pi}{2}}u)$. Find the homotopy group $\pi_1(X/\mathbb{Z})$ of the orbit space $X/\mathbb{Z}$.

7. Formulate and prove the Brower's Fixed Point Theorem (for a disk in the plane).

8. Let $\mathbb{P}^n$ be the projective space of lines $[x] = \mathbb{R}x$ in $\mathbb{R}^{n+1}$. We consider the map $f : \mathbb{P}^1 \to \mathbb{P}^2, x = [x_1, x_2] \mapsto [x_1, x_2, x_1 + x_2]$. Prove that $f$ is well-defined and find the homotopy group of $\mathbb{P}^2 \setminus f(\mathbb{P}^1)$.

8 problems, 24 point: 3 + 3 + 3 +3 +3 +3 +3+3. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

GZ