Mathematical Sciences (MV), GU
Thursday June 12, 2014, 8:30-12:30.

No aids (closed book, closed notes).
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Presence of teacher: $\sim 9:30$ and ~ 11.30

Exam in MMA 100 Topology, 7.5 HEC.

1. Let X be a topological space. Prove or disprove the following statements for subsets of X:

   (a) $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$. ($\overline{A}$ stands for the closure of A).

   (b) $A^o \cap B^o = (A \cap B)^o$. ($A^o$ stands for the subset of interior points of A).

2. We define three topological spaces as follows: $A = \mathbb{R}^2/\sim$ is the identification space by identifying the subset $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ as a single point and each other point of $\mathbb{R}^2$ as itself, $B \subset \mathbb{R}^3$ is the union of spheres of radius $\frac{1}{n}$ tangential to the xy-plane at the origin (a spherical earring), and $C \subset \mathbb{R}^3$ is the union of spheres of radius n tangential to the xy-plane at the origin. Find homeomeorphic and non-homeomorphic pairs among the spaces A, B and C. Provide brief geometric arguments.

3. Let (X, d) be a metric space. Prove that if $C \subset X$ is closed and $p \notin C$ then there exists a continuous function $f : X \to \mathbb{R}$ such that $f = 0$ on C, i.e. $f(c) = 0, \forall c \in C$, and $f(p) = 1$.

4. A continuous map $f : X \to Y$ is called *proper* is $f^{-1}(C)$ is compact for any compact set $C \subset Y$.

   [a] Prove that any proper map $f : \mathbb{R}^n \to \mathbb{R}^m$ is unbounded.

   [b] Suppose X is compact and Y is Haussdorff. Prove that any continuous map $f : X \to Y$ is proper.

5. Let $Func(\mathbb{R}^n)$ be the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$. Let $M_{C,U} = \{f \in Func(\mathbb{R}^n); f \text{ maps } C \text{ into } U\}$, where $C \subset \mathbb{R}^n$ is compact and $U \subset \mathbb{R}$ is open. We define a topology on $Func(\mathbb{R}^n)$ by requiring that the subsets $\{M_{C,U}\}_{C,U}$ form a basis. Prove that (**a**) $Func(\mathbb{R}^n)$ is Haussdorff, and (**b**) If $\lim_{n\to\infty} f_n = f$ in $Func(\mathbb{R}^n)$ then $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}^n$.

6. Prove that if X and Y have the same homotopy type then $\pi_1(X) = \pi_1(Y)$, i.e. isomorphic.

7. Let $\mathbb{P}^3$ be the projective space of lines $[x] = \mathbb{R}x$ in $\mathbb{R}^4$. We write any vector x in $\mathbb{R}^4$ as $x = (u, v)$, $u, v \in \mathbb{R}^2$ and consider the subset $X = \{[x] \in \mathbb{P}^3; x = (u, u),\ 0 \neq u \in \mathbb{R}^2\}$. Find the homotopy group of $\mathbb{P}^3 \setminus X$. (Hint: Consider the map $[u, v] \to [u - v] \in \mathbb{P}^1 = S^1$ and the homotopy $h(t, [u, v]) = [u - tv, (1 - t)v]$.)

8. Consider the map $f : \mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $f(u, v, w) = ((u \times v) \times w, (u \cdot v)w, (u \times v) \cdot w)$. Prove that f induces a well-defined map from $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^6$. Describe the induced map on the homotopy groups. (Reminder: $u \times v$ is the cross-product/vector-product in $\mathbb{R}^3$.)

8 problems, 24 point: 3 + 3 + 3 +3 +3 +3 +3+3. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

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