Solution. Exam in MMA 100 Topology, Thursday, June 11, 2015, 8:30-12:30.

1. (a) False: Example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1-|x|}{|x|+1}$ f maps the closed set $\mathbb{R}$ to the open but not closed set $(-1, 1)$ in $\mathbb{R}$.

   (b) False: Let X be the the real line equipped with the topology being defined by the basis (a, ∞), $a \in \mathbb{R}$. Then the set $[0, \infty)$ is compact but not closed.

2. Answer: X and Y are not homeomorphic, X is Hausdorff where as Y is not.

   X and Y are homotopic and have their fundamental group being $\mathbb{Z}$.

3. Proof: We prove that $f^{-1} : f(X) \to X$ is a map. Let $U \subset X$ be open, then U^c is closed and thus compact since X is compact. Thus $f(U^c)$ is compact and then closed since Y is Hausdorff. The inverse image of U under f^{-1} is $(f^{-1})^{-1}(U) = f(U)$, and its complement in $f(X)$ is $f(U^c)$ which is closed, namely $f(U)$ is open.

4. Solution: View the torus as the quotient group of $\mathbb{R}^2/\mathbb{Z}^2$ with the discrete group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by the group operation. Any discrete subgroup of the torus has its preimage being a discrete subgroup of $\mathbb{R}^2$ and vice versa. A discrete subgroup of $\mathbb{R}^2$ has its projection in $\mathbb{R}$ being discrete and is of the form $\mathbb{Z}\theta$, where θ is a rational number. Thus all discrete subgroups of the torus are of the form

   $$H_{(\theta_1,\theta_2)} = \{(e^{i2\pi\theta_1 n}, e^{i2\pi\theta_2 m}), n, m \in \mathbb{Z}\}$$

   where θ_1, θ_2 are rational numbers.

5. Solution: The action of $\mathbb{Z}^2$ on X can be written as the product of the action of $\mathbb{Z}$ on $\mathbb{C} \setminus \{0\}$ by

   $$m : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \quad w \mapsto e^m w$$

   with that of $\mathbb{Z}$ on $\mathbb{C} \times S^1$,

   $$n : \mathbb{C} \times S^1 \to \mathbb{C} \times S^1, \quad (z, s) \mapsto (e^{in\pi} z, e^{i\frac{2\pi}{3}n} s) = ((-1)^n z, \omega^n s), \omega = e^{i\frac{2\pi}{3}}$$

   The quotien space of the first one is a torus (thought as a ring with inner circle of radius 1 and outer circle of radius e being identified) and has its fundamental group $\mathbb{Z}^2$. So determine the second quotient space. The space $\mathbb{C}$ is contractive so we have a homotopy $F_t : \mathbb{C} \times S^1 \to \mathbb{C} \times S^1$,

   $$F_t(z, s) = (tz, s)$$

   from the trivial map $(z, s) \mapsto (0, s)$ for $t = 0$ to the identity map $(z, s) \mapsto (z, s)$ for $t = 1$. We claim that F_t induces a homotopy on the quotient space $\mathbb{C} \times S^1/\mathbb{Z} \to \mathbb{C} \times S^1/\mathbb{Z}$ from the trivial map $[(z, s)] \to [(0, s)]$ to the identity map $[(z, s)] \to [(0, s)]$. Indeed

   $$F_t(n(z, s)) = F_t((-1)^n z, \omega^n s) = (t(-1)^n z, \omega^n s) = nF_t((z, s));$$

   namely the action of n commutes with F_t. Thus

   $$[F_t] : \mathbb{C} \times S^1/\mathbb{Z} \to \mathbb{C} \times S^1/\mathbb{Z}, \quad [(z, s)] \mapsto [F_t(z, s)]$$

   is a well-defined map (and is continuous since $\mathbb{Z}$ acts discretely). In other words,

   the subset $\{0\} \times S^1/\mathbb{Z}$ is a homotopy retract in $\mathbb{C} \times S^1/\mathbb{Z}$.

   Thus the fundamental group of $\mathbb{C} \times S^1/\mathbb{Z}$ is the same as $\{0\} \times S^1/\mathbb{Z}$, which is $\mathbb{Z}$ since the latter is again a circle since it is covered by the circle three times.

6. Solution: Let

$$Y = \{[x_1, x_2] \in \mathbb{P}^{2n-1}; |x_1| = |x_2| = 1\} \subset X$$

Then

$$Y = S^{n-1} \times S^{n-1}/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts on $S^{n-1} \times S^{n-1}$ by $x \mapsto \pm x$. Consider the map

$$f : X \to X, \quad [x_1, x_2] \mapsto [\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}]$$

By definition of X we see that f is well-defined and

$$f : X \to Y = S^{n-1} \times S^{n-1}/\mathbb{Z}_2 \subset X$$

where Y is the quotient of the product of spheres with $\mathbb{Z}_2$ acting by sign change $(y_1, y_2) \to \pm(y_1, y_2)$. We claim that Y is a homotopy retract of X. Indeed let

$$F_t : f : X \to X, \quad [x_1, x_2] \mapsto [tx_1 + (1-t)\frac{x_1}{|x_1|}, tx_2 + (1-t)\frac{x_2}{|x_2|}]$$

F_t is well-defined and

$$F_1 = Id, \quad F_0 = f,$$

proving our claim. Thus the homotopy group of X is the same as $Y = S^{n-1} \times S^{n-1}/\mathbb{Z}_2$.

If $n > 2$ we have $\pi_1(Y) = \mathbb{Z}_2$ since S^{n-1} is simply connected.

If $n = 2$ we have $S^1 \times S^1$ is a group and $Y = S^1 \times S^1/\mathbb{Z}_2$ is quotient space by the subgroup $\{(1,1), (-1,-1)\}$, and is a group since $S^1 \times S^1$ is abelian. Now the fundamental group of any topological group is abelian, thus $\pi_1(Y)$ is abelian. Now $S^1 \times S^1$ is a covering of Y with covering index 2. Thus $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$ is a subgroup of the abelian group $\pi_1(Y)$ of index 2, $\pi_1(Y) = \pi_1(S^1 \times S^1) \cup \alpha\pi_1(S^1 \times S^1)$; the element α is representated by the path from $(1,1)$ to $(-1,-1)$ in $S^1 \times S^1$, whose square α^2 is represenatated by a loop in $S^1 \times S^1$. Thus we have $\pi_1(Y) = \mathbb{Z}^2$ (and the subgroup $\pi_1(S^1 \times S^1)$ is identified with $\mathbb{Z} \oplus 2\mathbb{Z}$ in $\mathbb{Z} \oplus \mathbb{Z}$.) (Alternatively the action of $\{\pm\}$ on $S^1 \times S^1$ can be realized via the homeomorphism $(x, y) \to (x, xy)$ as an action on the first factor S^1 and trivial on the second, thus $Y \sim_{homeo} S^1/\{\pm\} \times S^1 =\sim_{homeo} S^1 \times S^1$.)

See the textbook for solutions of Prob. 7-8.