Mathematical Sciences (MV), GU
Thursday, June 11, 2015, 8:30-12:30.

No aids (closed book, closed notes).
Phone: 0703-088304 Christoffere Standar
Presence of teacher: ~9:30 and ~11:30

Exam in MMA 100 Topology, 7.5 HEC.

1. Prove or disprove the following claims for general topological spaces: (a) A continuous map $f : X \to Y$ maps closed sets to closed sets. (b) A compact subset of a topological space is always closed.

2. Consider two families of circles $\{C_n\}$ and $\{D_n\}$ in the plane $\mathbb{E}^2$ with center 0: C_n has radius n, $n = 1, 2, \cdots$, while as D_n has radius $\frac{1}{n}$, $n = 1, 2, \cdots$. Let X (respectively Y) be the plane with all $\{C_n\}$ being identified with C_1 (respectively all D_n being identified with D_1). Are X and Y homeomorphic? Are X and Y homotopic?

3. Suppose X is a compact space and Y is a Hausdorff space. Prove that if $f : X \to Y$ is a one-to-one map then $f : X \to f(X)$ is a homeomorphism, where $f(X)$ is equipped with the induced topology.

4. Let $S^1 = \{e^{i\theta}; 0 \le \theta \le 2\pi\}$ be the circle group in $\mathbb{C}$ (with the usual product of complex numbers). The torus $S^1 \times S^1$ is a topological group equipped with the natural product, $(z, w)(z_1, w_1) = (zz_1, ww_1)$. Find all discrete subgroup of the torus. (Recall that a discrete subgroup is a subgroup with the induced topology being discrete.)

5. Let $X = \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \times S^1$, where $\mathbb{C}$ is the complex plane and S^1 the unit circle. Let $\mathbb{Z}^2$ act on X by the following: $(n, m) \in \mathbb{Z} : X \to X$, $(z, w, e^{it}) \mapsto (e^{in\pi}z, e^m w, e^{i\frac{2\pi}{3}n}e^{it})$. Find the fundamental group of the orbit space $X/\mathbb{Z}^2$.

6. Let $\mathbb{P}^{2n-1}$, $n > 1$, be the projective space of lines $[x] = \mathbb{R}x$ in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, $x \neq 0$. We consider the subset $X = \{[x_1, x_2]; x_1, x_2 \in \mathbb{R}^n, x_1 \neq 0, x_2 \neq 0\}$. Prove that X is path-connected and find the fundamental group of X. (The answer depends on n.)

7. Formulate the definition that $f : \tilde{X} \to X$ is a covering map. Prove that a covering map $f : \tilde{X} \to X$ induces an injective homomorphism of the fundamental groups $f_* : \pi_1(\tilde{X}, p) \to \pi_1(X, q)$, where $q = f(p)$. (Hint: Use the lifting lemma.)

8. Formulate and prove the Brouwer fixed point theorem.

8 problems, 24 point: 3 + 3 + 3 +3 +3 +3 +3+3. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

GZ