Mathematical Sciences (MV), GU
Thursday, March 17, 2016, 14:00-18:00.

No aids (closed book, closed notes).
Phone: Ext 5325 Tim Cardilin
Presence of teacher: ∼15:00 and ∼16:45

Exam in MMA 100 Topology, 7.5 HEC.

1. Prove or disprove the following claims for general topological spaces: (**a**) A continuous injective mapping $f : X \to Y$ maps open sets $U \subset X$ to open sets $f(U) \subset Y$. (**b**) Any connected component of a topological space X is open.

2. Suppose $C \subset X$ is a compact subset of a Haussdorff space X. Prove that C is closed.

3. Let A and B be the following subsets of the Euclidean plane $\mathbb{R}^2$: $A = \{(x, n) \in \mathbb{R}^2; x \in \mathbb{R}, n = 1, 2, \cdots\}$, and $B = \{(x, nx) \in \mathbb{R}^2; x \in \mathbb{R}, n = 1, 2, \cdots\}$. Let X (respectively Y) be the identification space of the plane $\mathbb{R}^2$ with the subset A being identified with the origin $o = (0, 0)$ (respectively B identified with o) and the rest of the points are themselvs. Prove that X and Y are not homeomorphic.

4. Let $X = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x \neq 0, x_1x_4 - x_2x_3 = 0\}$ be equipped with the subspace topology of the Euclidean space $\mathbb{R}^4$. Prove that X is path-connected.

5. Prove that the orthogonal group $O(3)$ is isomorphic to $SO(3) \times \mathbb{Z}_2$ as topological groups. Is $O(2)$ isomorphic to $SO(2) \times \mathbb{Z}_2$ as groups? (Recall $\mathbb{Z}_2 = \{\pm 1\}$.)

6. Let $\mathbb{Z}_3 = \{e^{\frac{k}{3}2\pi i}, k = 0, 1, 2\}$ be the cyclic group of order 3. Consider the action ρ of $\mathbb{Z}_3$ on the torus $T = S^1 \times S^1 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$ defined by $\rho(e^{\frac{k}{3}2\pi i}) : (z_1, z_2) \mapsto (e^{\frac{k}{3}2\pi i}z_1, e^{-\frac{k}{3}2\pi i}z_2)$. Let $X = T/\mathbb{Z}_3$ be the orbit space. Prove that X is homeomorphic to the torus and describe the induced group homomorphism $p_*\pi_1(T) = \mathbb{Z}^2 \to \pi_1(X) = \mathbb{Z}^2$ of the natural projection $p : T \to X$. (Hint: Use the homeomorphism $(z_1, z_2) \mapsto (z_1, z_1z_2)$ to "trivialize" the action)

7. Formulate the definition that $\tilde{X}$ is a covering space of X. Find all the path-connected covering space $\tilde{X}$ of the space $\mathbb{P}^{n-1} \times S^1$, $n \geq 3$, such that the fundamental group $\pi_1(\tilde{X})$ is a normal subgroup of $\pi_1(\mathbb{P}^{n-1} \times S^1)$.

8. Formulate the definition that two spaces X and Y have the same homotopy type. Prove that two spaces with the same homotopy type have the isomorphic homotopy groups.

8 problems, 24 point: 3 + 3 + 3 +3 +3 +3 +3+3. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

GZ