Solution, Exam 2016-03-17, MMA 100 Topology

1. Prove or disprove the following claims for general topological spaces: (**a**) A continuous injective mapping $f : X \to Y$ maps open sets $U \subset X$ to open sets $f(U) \subset Y$. (**b**) Any connected component of a topological space X is open.

   (a). False. Counter example: Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$, $U = (0, 4\pi)$ is open and its image $f(U) = [-1, 1]$ is not open.

   (b). False. Consider the set $\mathbb{Q}$ of rational numbers equipped with the Euclidean topology. Its connected components are the singletons $\{r\}$, none of them is open.

2. Suppose $C \subset X$ is a compact subset of a Haussdorff space X. Prove that C is closed.

   – See the text book for the proof.

3. Let A and B be the following subsets of the Euclidean plan $\mathbb{E}^2$: $A = \{(x, n) \in \mathbb{R}^2; x \in \mathbb{R}, n = 1, 2, \cdots\}$, and $B = \{(x, nx) \in \mathbb{R}^2; x \in \mathbb{R}, n = 1, 2, \cdots\}$. Let X (respectively Y) be the identification space of the plane $\mathbb{R}^2$ with the subset A being identified with the origin $o = (0, 0)$ (respectively B identified with o) and the rest of the points are themselvs. Prove that X and Y are not homeomorphic. Does there exist an injective continous map from Y to X?

   Proof: X is Haussdorff where as Y is not, therefore they are not homeomorphic. Generally, if some equivalence class $[z]$ in a topological space Z is not closed and has a limit point, then $Z/\sim$ is not Haussdorff. Let's give a concrete proof. Take the class $[a] = [(0, 1)] \in Y$ of $a = (0, 1)$. Then $[a] = a$ is identified with itself. Any neighborhood V of $[a] \in Y$ is represented by a neighborhood U in $\mathbb{E}^2$ of $(0, 1) \in \mathbb{E}^2$, $V = [U]$, which then contains a point $(\frac{1}{n}, 1)$ for n sufficiently large, namely V contains then the point $[(\frac{1}{n}, 1)] = [o]$, since $(\frac{1}{n}, 1)$ is identified with o. This proves that Y is not Haussdorff. To prove that X is Haussdorff we let $[p] \neq [q]$, $[p], [q] \in X$. Suppose none of them is $[o]$, i.e. they are not on the lines $y = n$. Then we can choose disjoint neighborhoods U_p and U_q of p and q respectively in $\mathbb{E}^2$ such that they have no intersection with the lines $y = n$. Then $[U_p] = U_p$ and $[U_q] = U_q$ are in disjoint neighborhood of $[p]$ and $[q]$ in X. Suppose $[p] = [o] = A$ and $[q] \neq [o]$, i.e. $q \notin A$. Then A is a closed set in the Euclidean space $\mathbb{E}^2$, we can choose a neighborhood U_q of q in $\mathbb{E}^2$ and V of A such that $U_q \cap V = \emptyset$. Thus $[U_q] = U_q$ and $[V] = V$ are neighborhoods of $[q]$ and $[o]$, and $[U_q] \cap [V] = \emptyset$.

4. Let $X = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4; x \neq 0, x_1x_4 - x_2x_3 = 0\}$ be equipped with the subspace topology of $\mathbb{E}^4$. Prove that X is path-connected.

   Proof: The set X can be interpreted as the set of pairs of parallell vectors $u = (x_1, x_2)$ and $u = (x_3, x_4)$ which are not vanishing simultaneously.) We fix a reference point $e_1 = (1, 0, 0, 0)$. Let $x = (x_1, x_2, x_3, x_4) = (u, v)$ be a general point.

Case 1: $u = (x_1, x_2) \neq 0$. We can first join x to a point $y = (y_1, y_2, y_3, y_4)$ where (y_1, y_2) is a point on the unit circle. Indeed $t \mapsto t(x_1, x_2, x_3, x_4)$ for t in the segment between 1 and $\frac{1}{\|u\|}$ is an arc joining x to y with $(y_1, y_2) = \frac{u}{\|u\|}$ is a point on the unit circle S^1. Thus $(y_3, y_4) = c(y_1, y_2)$ for some c. Now the circle S^1 is path-connected so there is a path $u(t), t \in [0, 1]$ joining $(1, 0)$ to (y_1, y_2), $u(0) = (1, 0)$, $u(1) = (y_1, y_2)$, consequently

$$t \mapsto (u(t), ctu(t)) \in X$$

is a path joining $e_1 = (1, 0, 0, 0)$ to $(u(1), cu(1)) = (y_1, y_2, y_3, y_4)$.

Case 2: $u = (x_1, x_2) = 0$. Then $v = (x_3, x_4) \neq 0$. We can make a path switching u and v. Indeed

$$t \mapsto (tx_3, tx_4, (1-t)x_3, (1-t)x_4), \quad t \in [0, 1]$$

is a curve in X joining $(0, 0, x_3, x_4)$ to the point $(x_3, x_4, 0, 0)$. This then reduces the Case 1 above.

5. Prove that the orthogonal group $O(3)$ is isomorphic to $SO(3) \times \mathbb{Z}_2$ as topological groups. Is $O(2)$ isomorphic to $SO(2) \times \mathbb{Z}_2$ as groups? (Recall $\mathbb{Z}_2 = \{\pm 1\}$.)

   Proof. Observe that the diagonal matrix $-I$ is in the center of $O(3)$. So the map

   $$h : O(3) \to SO(3) \times \mathbb{Z}_2; g \mapsto ((\det g)g, \det g)$$

   with inverse map

   $$h^{-1} : SO(3) \times \{\pm 1\} \to O(3); \quad (g, \pm 1) \mapsto (\pm 1)g$$

   is a group isomorphism and a topological homeomorphism since $\det g$ is continous.

   However $O(2)$ is not isomorphic to $SO(2) \times \mathbb{Z}_2$ since the center $O(2)$ is the group $\{\pm 1\}$ and $SO(2) \times \mathbb{Z}_2$ has center $\mathbb{Z}_2 \times \mathbb{Z}_2$.

6. Let $\mathbb{Z}_3 = \{e^{\frac{k}{3}2\pi i}, k = 0, 1, 2\}$ be the cyclic group of order 3. Consider the action ρ of $\mathbb{Z}_3$ on the torus $T = S^1 \times S^1 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$ defined by $\rho(e^{\frac{k}{3}2\pi i}) : (z_1, z_2) \mapsto (e^{\frac{k}{3}2\pi i}z_1, e^{-\frac{k}{3}2\pi i}z_2)$. Let $X = T/\mathbb{Z}_3$ be the orbit space. Prove that X is homeomorphic to a torus and describe the induced group homomorphism $p_*\pi_1(T) = \mathbb{Z}^2 \to \pi_1(X) = \mathbb{Z}^2$ of the natural projection $p : T \to X$. (Hint: Use the homeomorphism $(z_1, z_2) \mapsto (z_1, z_1 z_2)$ to "trivialize" the action)

   Solution: Consider the map $h : T \mapsto T, (z_1, z_2) \mapsto (z_1, z_1 z_2)$ and the group action λ of $\mathbb{Z}_3$ on T, $\lambda(e^{\frac{k}{3}2\pi i}) : (z_1, z_2) \mapsto (e^{\frac{k}{3}2\pi i}z_1, z_2)$. Then h is a homeomorphism and we have $h^{-1} \circ \lambda(g) \circ h = \rho(g)$ for $g \in \mathbb{Z}_3$. Thus the orbit space of the action of ρ is homeomorphic to the action of λ, which is $S^1/\mathbb{Z}_3 \times S^1$ and is further homeomorphic to $S^1 \times S^1 = T$. The induced homomorphism p_* is $\mathbb{Z}^2 \mapsto \mathbb{Z}^2 : (n, m) \mapsto (3n, m)$.

7. Formulate the definition that $\tilde{X}$ is a covering space of X. Find all the path-connected covering spaces $\tilde{X}$ of the space $\mathbb{P}^{n-1} \times S^1$, $n \geq 3$.

   Solution: The fundamental group $\pi_1(\mathbb{P}^{n-1} \times S^1)$ is $\mathbb{Z}_2 \times \mathbb{Z}$. Any subgroup of $\mathbb{Z}_2 \times Z$ is of the form $\mathbb{Z}_2 \times m\mathbb{Z}$, or $\{1\} \times m\mathbb{Z}$. In the first case the covering space is $\mathbb{P}^{n-1} \times S^1$ or $\mathbb{P}^{n-1} \times \mathbb{R}$ with the covering map $(p, s) \mapsto (p, s^m)$, or $(p, x) \mapsto (p, e^{2\pi ix}$. In the second case the covering space is $S^{n-1} \times S^1$ or $S^{\times}\mathbb{R}$ with the covering map $(p, s) \mapsto ([p], s^m)$, or $(p, x) \mapsto ([p], e^{2\pi ix}$, where $p \to [p]$ is the defining covering of $\mathbb{P}^{n-1}$ by S^{n-1}.

8. Formulate the definition that two spaces X and Y have the same homotopy type. Prove that two spaces with the same homotopy type have the isomorphic homotopy groups.

   – See the text book for the proof.

8 problems, 24 point: 3 + 3 + 3 +3 +3 +3 +3+3. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

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