Mathematical Sciences (MV), GU
Thursday, August 17, 2017, 8:30-12:30.

No aids (closed book, closed notes).
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Presence of teacher: ∼9:45 and ∼11:15

Solution. Exam in MMA 100 Topology, 7.5 HEC.

1. Prove or disprove the following claims: Let X be a topological space and $A, B \subset X$. (a) $(A \cap B)^0 = A^o \cap B^o$, where C^o stands for the set of inner points of C. (b) $f(A^0) = (f(A))^0$ for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and subset $A \subset \mathbb{R}$.

   Solution. (a) True. Proof: $A^o \cap B^o \subset A$, $A^o \cap B^o \subset B$, thus $A^o \cap B^o \subset A \cap B$. But $A^o \cap B^o$ is open, so $A^o \cap B^o \subset (A \cap B)^o$ by definition. Now if $x \in (A \cap B)^o$ then $x \in A \cap B$, and there exists a neighborhood N such that $N \subset A \cap B$, namely $N \subset A$, $N \subset B$, consequently $x \in A^o$, $x \in B^o$, and further $x \in A^o \cap B^0$.

   (b) False. Counter example: $f(x) = 1$, the constant function. Then $f(\mathbb{R}^0) = f(\mathbb{R}) = 1$ and $f(\mathbb{R})^o = \emptyset$.

2. Suppose X is a compact topological space and $f : X \to \mathbb{R}$ is a map (i.e. a continuous function). Prove that there exist $x_0, x_1 \in X$, such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.

   Proof. $f : X \to \mathbb{R}$ is continuous and X is compact, so is $f(X) \subset \mathbb{R}$ a compact subset of $\mathbb{R}$. A compact subset in $\mathbb{R}$ is closed and bounded, so $\sup f(X)$ and $\inf f(X)$ exist and are in $f(X)$. Namely there exists $x_0, x_1 \in X$ such that $f(x_0) = \inf f(X)$, $f(x_1) = \sup f(X)$, equivalently $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.

3. Suppose (X, d) is a metric space and fix $x_0 \in X$ and $\delta > 0$. Prove that there exist a continous function $f : X \mapsto [0, 1]$, such that $f(x) = 1$ for $d(x, x_0) \leq \delta$ and $f(x) = 0$ for $d(x, x_0) \geq 2\delta$.

   Proof. Consider the subset $A \cup B$ where

   $$A = \{x \in X; d(x, x_0) \leq \delta\}, \quad B = \{x \in X; d(x, x_0) \geq 2\delta\}.$$

   Then A and B are closed and $A \cap B = \emptyset$. Let $g : A \cup B \to [0, 1]$, $g = 1$ on A and $g = 0$ on B. g is continuous and $0 \leq g \leq 1$. It follows from Tietze extension theorem that there exists $f : X \to [0, 1]$ extending g.

4. Let $\mathbb{R}^3$ be the Euclidean space with the standard basis $\{e_1, e_2, e_3\}$. Let $X = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3; (u, v, e_3)$ forms a basis of $\mathbb{R}^3\}$. Prove that X has two path-connected components.

   Proof. Observe first that the 3×3-matrix (u, v, e_3) is a basis iff the matrix (u, v, e_3) has determinant non-zero, the matrix (u, v, e_3) is of the form $(u, v, e_3) = \begin{pmatrix} A & 0 \\ * & 1 \end{pmatrix}$ and $\det(u, v, e_3) = \det A$. Each 2×2-matrix A can be written as $A = OP$ where O is an orthogonal matrix and P is an upper triangular matrix with positive diagonal elements, $P = \begin{pmatrix} e^{t_1} & p_{12} \\ 0 & e^{t_2} \end{pmatrix}$. Now each P can be connected to I by a path and each O can be connected to the identify $I = (e_1, e_2)$ if $\det O = 1$ or to (e_2, e_1) if $\det O = -1$. This proves the claim.

5. Let $0 < r_1, r_2 < 1$ be two fixed real numbers. Consider the following action of $\mathbb{Z}$ on the torus $\mathbb{T}^2 = T \times T = \{(e^{i\theta_1}, e^{i\theta_2}), 0 \le \theta_1, \theta_2 < 2\pi\}$.

$$n \in \mathbb{Z} : (e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_1 + inr_1}, e^{i\theta_2 + inr_2}).$$

(a) Find r_1, r_2 so that the orbit space $\mathbb{T}^2/\mathbb{Z}$ is Haussdorff. (b) Find r_1, r_2 so that any orbit $[(e^{i\theta_1}, e^{i\theta_2})] = \mathbb{Z}(e^{i\theta_1}, e^{i\theta_2})$ of $\mathbb{Z}$ is dense in $\mathbb{T}^2$.

Proof. We write $r_1 = 2\pi x_1, r_2 = 2\pi x_2$. (a) Recall that x is irrational iff $\{e^{2\pi inx}\}$ is dense on the unit circle. Thus the orbit space space is Hausdorff iff both x_1 and x_2 are rational.

(b) If $\{x_1, x_2, 1\}$ generate a three dimensional space over $\mathbb{Q}$, in other words, x_1 and x_2 are not related by $x_1 = px_2 + q$, nor $x_2 = px_1 + q$ for some rational numbers p and q, then the orbit subgroup $\mathbb{Z}(x_1, x_2) + \mathbb{Z}e_1 + \mathbb{Z}e_2$ is dense in $\mathbb{R}^2$, equivalently the set $\{(\{nx_1\}, \{nx_2\})\}$ is dense in the unit square I^2, also equivalently the above action of $\mathbb{Z}$ is dense on the torus. (This is part of the Kronecker's theorem. No proof is required for the answer.)

6. Let $S = \{c \in \mathbb{C}; |c| = 1\}$ be the circle in $\mathbb{C}$ and $S^3 = \{(c_1, c_2) \in \mathbb{C}^2; |c_1|^2 + |c_2|^2 = 1\}$ be the Euclidean 3-sphere in $\mathbb{C}^2$. Consider the following action of the cyclic group $\mathbb{Z}_n = \{e^{2\pi i\frac{j}{n}}, j = 0, \cdots, n-1\}$ on $S \times S^3$, $e^{2\pi i\frac{j}{n}} : (c, c_1, c_2) \mapsto (e^{2\pi i\frac{j}{n}}c, e^{2\pi i\frac{j}{n}}c_1, e^{2\pi i\frac{j}{n}}c_2)$. Find the fundamental group of the orbit space $S \times S^3/\mathbb{Z}_n$.

Solution. Call τ the action. We take a homeomorphism of $S \times S^3$ to "trivialize" the action τ of $\mathbb{Z}_n$. Let $f : S \times S^3 \to S \times S^3$, $(c, q) \mapsto (c, c^n q)$. Then the action $f \circ \tau \circ f^{-1}$ is $\epsilon = e^{2\pi i\frac{j}{n}} : (c, q) \mapsto (\epsilon c, q)$. Thus the quotient space is $S \times S^3/\mathbb{Z}_n = (S/\mathbb{Z}_n) \times S^3$, and the fundamental group is $\pi_1(S/\mathbb{Z}_n) \times \pi_1(S^3) = \mathbb{Z} \times 1 = \mathbb{Z}$.

7. Let $\mathbb{P}^5$ be the projective space of lines $[x, y] = \mathbb{R}(x_1, x_2, x_3, y_1, y_2, y_3)$ in $\mathbb{R}^6$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Let $X = \{[x, y] \in \mathbb{P}^5; x \not\parallel y\}$. Find the fundamental group of X. (Here $x \not\parallel y$ means that x is not parallel to y; the zero vector is considered parallel to any vector.)

Sol: (This is somewhat related to Problem 4 above). It follows from the definition above that $[x, y] \in X$ if and only if $x \ne \lambda y$ and $y \ne \lambda x$, in particular $x \ne 0, y \ne 0$. Hence $X = \{[x, y] \in \mathbb{P}^5; x, y \in \mathbb{R}^3, \{x, y\} \text{ is linear independent}\}$. Now by the Gram-Schmidt orthogonalization procedure each pair (x, y), viewed as 3×2-matrix, can be written as $(x, y) = (u, v)A$ where (u, v) is a pair of orthonormal vectors and A is 2×2 upper-diagonal matrix with positive diagonal elements. The set of matrices A with the above property is homotopy to the single set $\{I_2\}$ of identity matrix (see the lecture notes), thus X is homotopy equivalent to the set

$$\{[u,v] \in \mathbb{P}^5; u, v \in \mathbb{R}^3, (u,v) \quad \text{is orthonormal}\},$$

which is further more the set

$$Y/\mathbb{Z}_2, \quad Y = \{(u,v) \in S^2 \times S^2; (u,v) \quad \text{is orthogonal}\},$$

where $\mathbb{Z}_2$ is acting on Y by $(u,v) \to \pm(u,v)$, namely it is double-covered by Y. We determine first $\pi(Y)$.

The set Y is identified with the group $SO(3)$ by the correspondence $(u, v, u \times v)$. Thus $\pi(Y) = \mathbb{Z}_2$.

Consider next the injective group homomorphism $\pi_1(Y) \to \pi_1(Y/\mathbb{Z}_2)$ induced by the covering, viewed as a subgroup inclusion. The coset space $\pi_1(Y/\mathbb{Z}_2)/\pi_1(Y)$ has cardinality 2 since it's a double cover. Hence $\pi_1(Y/\mathbb{Z}_2)$ has $\pi_1(Y) = \mathbb{Z}_2$ as a normal subgroup and $\pi_1(Y/\mathbb{Z}_2) = \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Finally the simple curve in Y from (e_1, e_2) to $(-e_1, -e_2)$ induces a non-trivial loop in $Y/\mathbb{Z}_2$ whose square is the element -1 in $\pi_1(Y)$. Thus $\pi_1(Y/\mathbb{Z}_2) = \mathbb{Z}_4$.

8. Formulate and prove the Brouwer fixed point theorem for mappings of the closed unit disc.

   See the textbook.

8 problems, 24 points $= 8 \times 3$. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

GZ