Mathematical Sciences (MV), GU
Thursday, August 17, 2017, 8:30-12:30.

No aids (closed book, closed notes).
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Presence of teacher: ∼9:45 and ∼11:15

Exam in MMA 100 Topology, 7.5 HEC.

1. Prove or disprove the following claims: Let X be a topological space and $A, B \subset X$. (a) $(A \cap B)^0 = A^o \cap B^o$, where C^o stands for the set of inner points of C. (b) $f(A^0) = (f(A))^0$ for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and subset $A \subset \mathbb{R}$.

2. Suppose X is a compact topological space and $f : X \to \mathbb{R}$ is a map (i.e. a continuous function). Prove that there exist $x_0, x_1 \in X$, such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.

3. Suppose (X, d) is a metric space and fix $x_0 \in X$ and $\delta > 0$. Prove that there exist a continous function $f : X \mapsto [0, 1]$, such that $f(x) = 1$ for $d(x, x_0) \leq \delta$ and $f(x) = 0$ for $d(x, x_0) \geq 2\delta$.

4. Let $\mathbb{R}^3$ be the Euclidean space with the standard basis $\{e_1, e_2, e_3\}$. Let $X = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3; (u, v, e_3)$ forms a basis of $\mathbb{R}^3\}$. Prove that X has two path-connected components.

5. Let $0 < r_1, r_2 < 1$ be two fixed real numbers. Consider the following action of $\mathbb{Z}$ on the torus $\mathbb{T}^2 = T \times T = \{(e^{i\theta_1}, e^{i\theta_2}), 0 \leq \theta_1, \theta_2 < 2\pi\}$.
$$n \in \mathbb{Z} : (e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_1 + inr_1}, e^{i\theta_2 + inr_2}).$$
(a) Find r_1, r_2 so that the orbit space $\mathbb{T}^2/\mathbb{Z}$ is Haussdorff. (b) Find r_1, r_2 so that any orbit $[(e^{i\theta_1}, e^{i\theta_2})] = \mathbb{Z}(e^{i\theta_1}, e^{i\theta_2})$ of $\mathbb{Z}$ is dense in $\mathbb{T}^2$.

6. Let $S = \{c \in \mathbb{C}; |c| = 1\}$ be the circle in $\mathbb{C}$ and $S^3 = \{(c_1, c_2) \in \mathbb{C}^2; |c_1|^2 + |c_2|^2 = 1\}$ be the Euclidean 3-sphere in $\mathbb{C}^2$. Consider the following action of the cyclic group $\mathbb{Z}_n = \{e^{2\pi i \frac{j}{n}}, j = 0, \cdots, n-1\}$ on $S \times S^3$, $e^{2\pi i \frac{j}{n}} : (c, c_1, c_2) \mapsto (e^{2\pi i \frac{j}{n}} c, e^{2\pi i \frac{j}{n}} c_1, e^{2\pi i \frac{j}{n}} c_2)$. Find the fundamental group of the orbit space $S \times S^3/\mathbb{Z}_n$.

7. Let $\mathbb{P}^5$ be the projective space of lines $[x, y] = \mathbb{R}(x_1, x_2, x_3, y_1, y_2, y_3)$ in $\mathbb{R}^6$, where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Let $X = \{[x, y] \in \mathbb{P}^5; x \not\parallel y\}$. Prove that X is path connected and find its fundamental group. (Here $x \not\parallel y$ means that x is not parallel to y; the zero vector is considered parallel to any vector.)

8. Formulate and prove the Brouwer fixed point theorem for mappings of the closed unit disc.

8 problems, 24 points $= 8 \times 3$. Grade limits: 12p for Godkänd (Pass), 18p for Väl Godkänd (Very Good).

GZ