Solution. Exam in MMA 100 Topology, 2017-03-16

1. (a) False. Counter examples:

   $A = \mathbb{Q} \subset \mathbb{R}$, $A^o = \emptyset$, $(\bar{A})^o = \mathbb{R}^o = \mathbb{R}$;

   $A = (0,1) \cup (1,2) \subset \mathbb{R}$, $A^o = A$, $(\bar{A})^o = (0,2)$.

   (b) See the text book. (Use definition of compactness and Hausdorff to prove that $X \setminus C$ is open.)

2. See the text book.

3. Let $p \neq q$ be two different points in (X, d). Let

   $$f(x) = \frac{d(x,p)}{d(x,p) + d(x,q)}$$

   Then f is well-defined, continuous $f(p) = 0, f(q) = 1$ and $0 \leq f(x) \leq 1$. But X is connected so $f(X)$ is a connected subset in $[0,1]$ containing $0, 1$. Thus $f(X)$ is the interval $f(X) = [0,1]$, i.e. f is onto.

4. (a) A discrete subgroup is a topological group with the discrete topology. Now if G contains a rotation $R = 2\pi\theta$ with irrational θ. Then the subgroup generated by R is dense in $SO(2)$ and thus G is not discrete.

   (b) If π is a covering then we can take a small unit disc U near 0 and take its image $\hat{U} = \pi(U)$ as a neighborhood of $\hat{0} \in Y = \mathbb{R}^2/G$. (See also the exercise 8 below.) Suppose G contains a reflection r or a (non-trivial) rotation $2\pi\theta$. Namely we have the following two cases:

   (1) G contains two different elements, the rotation r and the identity element 1, all fixing 0. For simplicity we can assume r is reflection in x-axis.

   (2) G contains a rotation θ, and by (a) above, θ is of the form $\theta = 2\pi\frac{q}{p}$, with $gcd(p,q) = 1$, $p > 1$. Thus G cotains all rotations by $\frac{j}{p}(2\pi)$ for $j = 0, \cdots, p-1$ (by the Euclidean algorithm).

   In the first case (1) any two points (x, y) and $(x, -y)$ are identified, and the preimages of $\hat{U}$ under $\pi : \mathbb{R}^2 \to Y$ contains the disc U, and π restricted to U can not be a homeomophism, since it is 2 to 1 (mapping two points to one) for any smaller neighborhood U. This is a contradiction. In the second case (2) π restricted to the neighborhood U of 0 maps p different points to one point except the orgin, again a contradiction.

5. We find a homotopy retract from X to X_0. Consider the map

$$H : X \times I \to X : H(c_1, c_2; t) = (c_1 - (1-t)c_2, tc_2).$$

Then H is continuous and

$$\underline{t=0}: \quad H(c_1, c_2; 0) = (c_1 - c_2, 0) \in X_0, (c_1, c_2) \in X; \quad H((c, 0); 0) = (c, 0), \quad (c, 0) \in X_0,$$

and

$$\underline{t=1}: \quad H(c; 0) = c,$$

the identity map on X. We have then X_0 is a homotopy retract of X and $\pi_1(X) = \pi_1(X_0) = \mathbb{Z}$. (More precisely

$$X_0 \hookrightarrow X \to X_0,$$

and

$$X \to X_0 \hookrightarrow X$$

are homotopy to the respective identity maps; see the text book)

6. Take $-p \notin f(X)$. Let

$$H(x, t) = \frac{tf(x) + (1-t)p}{\|tf(x) + (1-t)p\|}$$

f is well-defined: If $tf(x) + (1-t)p = 0$ for some x then $tf(x) = -(1-t)p$. Taking norm we see $t = \frac{1}{2}$ which in turn implies $\frac{1}{2}f(x) + \frac{1}{2}p = 0$, i.e. $f(x) = -p$ a contradiction. H is continous and $H(x, 0) = e_{-p}$, and $H(x, 1) = f(x)$.

7. If $[u, v] \in \mathbb{P}^1$ then one of u and v is not zero. Thus the vector $f(u, v) = (u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ is a nonzero vector and $[f(u, v)]$ represents a line. f is bilinear, $f(\lambda u, v) = \lambda f(u, v)$, $f(u, \lambda v) = \lambda f(u, v)$. Thus the induced map of f is well-defined.

   Fix referece points $p = [(1, 0)]$ on $\mathbb{P}^1$ and $q = [(1, 0, 0, 0)]$ on $\mathbb{P}^3$. A half unit circle C in $\mathbb{R}^2$ from p to $-p = (-1, 0)$ defines a generator $[C]$ of $\pi_1(\mathbb{P}^1) = \mathbb{Z} = \mathbb{Z}[C]$. f maps the half circle $C \times \{p\}$ to a half circle in $\mathbb{R}^4$ from q to $-q$, which generate the group $\pi_1(\mathbb{P}^3) = \{\pm 1\}$, i.e., -1; similarly for the half circle $\{p\} \times C$. Thus

$$f_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2, (m[C], n[C]) \mapsto (-1)^{m+n},$$

   since f_* is a group homomorphism.

8. See the text book for the definition.

   Let π be a covering. So for any $y \in Y$ there is a neighborhood $V = V_y$ of y such that $\pi^{-1}(V)$ is a disjoint union of open sets U_α in X and $\pi|_{U_\alpha}$ is a homeomorphism to V. Let

U be an open set in X, and write the map π as $y = \pi(x)$, $x \in U$. There is a neighborhood $V = V^x$ of $y = \pi(x)$ as above such that $\pi^{-1}(V^x) = \cup U_\alpha$. Thus $x \in U_\alpha$ for some $\alpha(x)$ (choose any), and $\pi : U \cap U_{\alpha(x)}$ is a homemorphism, in particular the image is open. Now $U = \cup_{x \in U}(U \cap U_{\alpha(x)})$ and its image is

$$\pi(\cup_{x \in U}(U \cap U_{\alpha(x)})) = \cup_{x \in U}\pi(U \cap U_{\alpha(x)})$$

which is open.

π is generally not a closed map: Take π the covering of $\mathbb{R}$ to the unit circle, $x \mapsto e^{2\pi x}$. Let $C = \cup_{n=1}^{\infty}\{n + \frac{1}{n}\}$. Then C is closed but its image is not.