Solution. Exam Thursday, March 15, 2018, MMA 100 Topology, 7.5 HEC.

1. Prove or disprove the following claims: (a) $(\overline{A})^0 = A^o$ for any subset $A \subset X$ of a topological space X, where $\overline{A}$ stands for the closure of A and A^o stands for the set of inner points of A. (b) $f(\overline{A}) = \overline{f(A)}$ for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and subset $A \subset \mathbb{R}$. (c) A covering map $f : X \to Y$ is an open map, i.e., f maps open sets to open sets.

   Solution. (a) False. Example: $A = \mathbb{Q} \subset \mathbb{R}$, $\bar{A} = \mathbb{R}$, $(\bar{A})^o = \mathbb{R}$, $A^0 = \emptyset$.

   (b) False. Ex. $f(x) = \arctan x$, Take $A = \mathbb{R}$, $A = \bar{A}$, $f(\bar{A}) = f(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\overline{f(A)} = [-\frac{\pi}{2}, \frac{\pi}{2}]$, which is not $f(\bar{A})$. (Or $f(x) = \frac{1}{1+|x|}$, $A = \mathbb{R}$, $f(\mathbb{R}) = (0, 1]$)

   (c) True. Take U an open set of X. Each $x \in U$ has a neighborhood $U_x \subset U$ for which $f : U_x \to f(U_x)$ is a homeomorphism, in particular $f(U_x)$ is open. Thus $U = \cup_{x \in U} U_x$ and $f(U) = f(\cup_x U_x) = \cup_x f(U_x)$ is also open.

2. Let $f : X \to Y$ be an injective onto map where X is compact and Y is Hausdorff. Prove that f is a homeomorphism. Provide a counter example to the claim when X is not compact.

   Solution. See the textbook for the proof. Counter ex: $f : (0, 2\pi] \to S^1$, $f(x) = e^{ix}$.

3. Suppose (X, d) is a connected metric space and X contains at least two points. Prove that there exists an *onto* continous function $f : X \mapsto [0, 1]^2$ to the unit square $[0, 1]^2$.

   Proof. Let $p \neq q$ be two different points of X. Let $f(x) = \frac{d(x,p)}{d(x,p)+d(x,q)}$. Then f is well-defined, continous, $0 \leq f(x) \leq 1, f(p) = 0, f(q) = 1$. X is connected thus $f(X)$ is connected and consequently f is onto, $f(X) = [0, 1]$. Now let $g : [0, 1] \to [0, 1]^2$ be a Peano curve, i.e, onto map. Then $g \circ f$ is onto $[0, 1]^2$.

4. Recall that if a topological group G acts on a space X and $H \subset G$ is a subgroup then there is a natural map $X/H \to X/G, Hx \mapsto Gx$ mapping H-orbits Hx to G-orbits Gx. Now the Klein's bottle K is an orbit space $K = \mathbb{R}^2/G$ by discrete group G of Euclidean motions. Describe one covering $K_1 = \mathbb{R}^2/H_1 \to K = \mathbb{R}^2/G$ which is normal (i.e., H_1 is a normal subgroup) and one $K_2 = \mathbb{R}^2/H_2 \to K = \mathbb{R}^2/G$ which is not normal.

   Solution. The Klein's bottle can be realized as a quotient space $\mathbb{R}^2/G$ of the plane $\mathbb{R}^2$ by the group G of the Euclidean motions generated by the translation T and the "reflected translation" S:

   $$T : (x, y) \mapsto (x + 1, y),\ S : (x, y) \mapsto (-x, y + 1).$$

   They satisfy the relation $S^{-1}TS = T^{-1}$, or equivalently $TS^{-1}TS = 1$. Let $H_1 = \langle T \rangle$ and $H_2 = \langle S \rangle$ be the subgroup generated by T and respectively by S. It follows from $S^{-1}TS = T^{-1}$ and $T^{-1}ST = ST^2$ (please check!) that H_1 is normal subgroup whereas H_2 is not. Consequently we have a normal covering $\mathbb{R}^2/H_1 \to \mathbb{R}^2/G$ and an non-normal one $\mathbb{R}^2/H_2 \to \mathbb{R}^2/G$.

5. Denote $\mathcal{M} = Map(X, Y)$ the set of all mappings from X to Y. We say that a subset $\mathcal{M}$ is path-connected if for any two maps $f_0, f_1 \in \mathcal{M}$, there is a homotopy $F : X \times [0, 1] \to Y$ of f_0 and f_1, i.e., $F(\cdot, 0) = f_0(\cdot)$, $F(\cdot, 1) = f_1(\cdot)$. Determine if the following two sets are path-connected and provide arguments for your claim. (a) $\mathcal{M} = Map(\mathbb{P}^2, \mathbb{P}^2)$. (b) $\mathcal{M} = Map(\mathbb{R}^2 \setminus \{0\}, S^1)$. ($S^1$ is the unit circle.) (Hint: Use fundamental groups.)

   Proof. Both spaces are not path-connected. Two maps $f, g \in \mathcal{M}$ are connected by a path is equivalent that they are homotopic. Thus they induces the same group homomorphisms $f_* = g_* : \pi_1(X) \to \pi_1(Y)$. (a) The fundamental group $\pi_1(\mathbb{P}^2) = \mathbb{Z}_2$. We take $f = Id$ the identity map and $g = e_p : x \to p$ the trivial map (for any fixed p). The corresponding group homomorphisms are $f_* = Id$ and $g_* = e : \{\pm\} \to 1$, and are not equal. (b) We may take $f(x) = x/|x|$, and $g(x) = e_p$ and get $f_*, g_* : \mathbb{Z} = \pi_1(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{Z} = \pi_1(S^1)$, $f_* = Id$ and the trivial map $g_* = e$.

6. Let $SL_n(\mathbb{R})$ be the group of real $n \times n$-matrices of determinant 1, $SL_n(\mathbb{R}) = \{T \in M_{n,n}; \det T = 1\}$. Prove that the subgroup $SO(n) = \{T \in M_{n,n}; T^tT = I, \det T = 1\}$ is a homotopy retract of $SL_n(\mathbb{R})$, i.e., there exists f such that the maps $SO(n) \xhookrightarrow[inclusion]{} SL_n(\mathbb{R}) \underset{f}{\hookrightarrow} SO(n)$ define spaces of the same homotopy type.

   Proof. Each matrix $T \in SL_n(\mathbb{R})$ represents an oriented basis in $\mathbb{R}^n$ and vice versa. The Gram-Schmidt orthogonalization states that each T can be written uniquely as $T = SU$, where S is an orthogonal matrix and U is an upper-triangular matrix with positive diagonal elements (the normalization constants). Furthermore $\det T = 1$ implies then $\det S > 0$ and consequently $\det S = 1, S \in SO(n)$. Let $f : SL_n \to SO(n), f(T) = S$, then f is continuous since Gram-Schmidt process is done by linear combintation of matrix elements of T and dividing the norms, all being continous functions in T. The following defines then the homotopy retract of $SO(n)$ in SL_n (we write the formula for 3×3-matrices, the general case is similar):

$$F(T, t) = SU(t),\, T = SU,\, U = \begin{pmatrix} e^{a_1} & * & * \\ 0 & e^{a_2} & * \\ 0 & 0 & e^{a_3} \end{pmatrix},\, U(t) = \begin{pmatrix} e^{a_1 t} & t* & t* \\ 0 & e^{a_2 t} & t* \\ 0 & 0 & e^{a_3 t} \end{pmatrix}.$$

   (Here $*$ represents the matrix element in that position.) Then $f \circ \iota = Id$ and $\iota \circ f \sim Id$ via the homotopy F.

7. Consider the set $X = \{([\mathbf{u}], [\mathbf{v}]) \in \mathbb{P}^2 \times \mathbb{P}^2; 0 \neq \mathbf{u} \in \mathbb{R}^3, 0 \neq \mathbf{v} \in \mathbb{R}^3, \mathbf{u} \not\parallel \mathbf{v}\}$ and the cross product $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \times \mathbf{v}$ in $\mathbb{R}^3$. Prove that the cross product induces a well-defined map $f : ([\mathbf{u}], [\mathbf{v}]) \mapsto [\mathbf{u} \times \mathbf{v}]$ from X to $\mathbb{P}^2$. Find the fundamental group of X and describe the group homomorphism $f_* : \pi_1(X) \to \pi_1(\mathbb{P}^2) = \mathbb{Z}_2$.

   Solution: (This problem is a bit more difficult than normal). First we represent all elements in $\mathbb{P}^2$ by unit vectors $\mathbf{u}$, namely $\mathbb{P}^2 = S^2/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting by ± 1, and consider the space Y

of non-parallell unit vectors $(\mathbf{u},\mathbf{v})$. By Gram-Schmidt orthogonalizaion procedure each pair $(\mathbf{u},\mathbf{v})$ corresponds to a pair of orthogonal vectors $(\mathbf{u},\mathbf{v}')$ with $\mathbf{v}' = \frac{\mathbf{v}+c\mathbf{u}}{\|\mathbf{v}+c\mathbf{u}\|}$, and $c = -\frac{(\mathbf{v},\mathbf{u})}{(\mathbf{u},\mathbf{u})}$. By changing c to tc, $0 \leq t \leq 1$ we see that our set Y is homotopic to the set of $(\mathbf{u},\mathbf{v})$ orthogonal unit vectors and X is homotopic to the quotient space $Y/\mathbb{Z}_2^2$,

$$X \sim_{homotopy} Y/\mathbb{Z}_2^2$$

Now each pair $(\mathbf{u},\mathbf{v})$ corresponds to an orientated ON-frame $(\mathbf{u},\mathbf{v},\mathbf{u}\times\mathbf{v})$ of $\mathbb{R}^3$, vice versa, each orientated ON-frame is of this form. In other words, the set Y is the orthogonal group $SO(3)$. Now $\mathbb{P}^2 \times \mathbb{P}^2 = S^2/\mathbb{Z}_2 \times S^2/\mathbb{Z}_2 = S^2 \times S^2/\mathbb{Z}_2^2$. Thus

$$X \sim_{homotopy} Y/\mathbb{Z}_2 \times \mathbb{Z}_2 \sim_{homeomorphic} SO(3)/\mathbb{Z}_2^2$$

with the action of $\mathbb{Z}_2^2$ on $SO(3)$ by $(\mathbf{u},\mathbf{v},\mathbf{u}\times\mathbf{v}) \mapsto (\pm\mathbf{u},\pm\mathbf{v},\pm\mathbf{u}\times\mathbf{v})$, changing the sign of exactly two vectors (to keep the orientation). Now the fundamental group of $SO(3)$ is $\mathbb{Z}_2$ (the same as $\mathbb{P}^3$) and its universal covering group is the group S^3 of unit quaternionic numbers $\mathbb{H}$, the double covering map is $u \mapsto (u_* : \mathbb{R}^3 = \Im H \to \Im H, x \to uxu^{-1})$, each u corresponding to a $SO(3)$ matrix $u_* = (uiu^{-1}, uju^{-1}, uku^{-1})$ with (ijk) being the standard frame in $\Im H$. The above action of $\mathbb{Z}_2$ lifts to S^3 producing the group K generated by by multiplication by the quaternionic unit on u, $u \mapsto u(\pm i), u(\pm j), u(\pm k)$ and $SO(3)/\mathbb{Z}_2^2$ being the orbit space,

$$X \sim_{homotopy} S^3/K.$$

Thus the fundamental group of X and $SO(3)/\mathbb{Z}_2^2$ is $\{\pm 1, \pm i, \pm j, \pm k,\}$, the Cayley/Hamiltonian group with 8 quaternionic units.

8. Formulate and prove the Brouwer's fixed point theorem for the *closed unit square* $[0,1]^2$ in $\mathbb{R}^2$.

   See the textbook. The same proof of the closed disc works also for the square.