Mathematical Sciences (MV), GU
June 10, 2019, 8:30-12:30.

No aids (closed book, closed notes).
Phone: Jeffrey Steif 070-2298318
Presence of teacher: $\sim 9:15$, $\sim 10:30$ and $\sim 11:30$

# Exam in MMA 100 Topology, 7.5 HEC.

8 problems. 24 points= $8 \times 3$. Grade limits: 12p for Godkänd (Pass). 18p for Väl Godkänd (Very Good). Each problem is graded as a whole; so it is not necessarily the case that if a problem has 4 parts then each part is worth $3/4$ of a point.

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1. Answer true or false to each of the following questions. No explanation is required.

   **[a]** A compact subset of a topological space is always closed.

   **[b]** If $X$ is a compact metric space and $A_1, A_2, \ldots$ is a sequence of closed sets, then $\cup_i A_i$ is a closed set.

   **[c]** If $X$ and $Y$ are connected, then $X \times Y$ is connected.

   **[d]** If $X$ and $Y$ are compact, then $X \times Y$ is compact.

   **[e]** If $X$ and $Y$ are Hausdorff, then $X \times Y$ is Hausdorff.

2. Show that if $f$ is a mapping from $R^n$ to $R^n$, then $f$ is continuous (meaning inverse images of open sets are open) if $f$ is continuous in the usual sense (meaning that for every $x$ and $\epsilon > 0$, there exists $\delta > 0$ so that $\|y - x\| < \delta$ implies that $\|f(y) - f(x)\| < \epsilon$).
   $\|x - y\|$ means the usual distance between $x$ and $y$.

3. **[a]** Give an example of a Hausdorff topological space $X$, a subset $A$ of $X$ and a point $x$ in the closure of $A$ such that there does not exist a sequence of elements in $A$ which converges to $x$. Prove your example works.

   **[b]** Does there exist such an example which is also a metric space?

4. **[a]** State the definition that a topological space is connected.

   **[b]** Does there exist a connected topological space $X$ and a continuous function $f$ from $X$ to the real numbers such that there exist $x_0, x_1 \in X$ with $f(x_0) = 11$, $f(x_1) = 32$ but no $x \in X$ with $f(x) = 23$? Justify your answer in either case.

   **[c]** Let $N$ be the positive integers and consider $R^N$ with metric $d(x, y) = \sup\{\min\{|x_i - y_i|, 1\} : i = 1, 2, 3, \ldots\}$. Is this space connected? Justify your answer.

5. Consider the set $[0, 1]^N$ in the product topology, the uniform topology and the box topology.

   **[a]** Which of these are compact? Justify your answer.

   **[b]** Let $A$ be the set of elements in $[0, 1]^N$ which are eventually 0. What is the closure of $A$ in these three different topologies. You do NOT need to justify your answer.

   **[c]** Which of these spaces are metrizable? You do NOT need to justify your answer.

6. Define the quotient topology and prove that the quotient space obtained from the unit interval $[0, 1]$ where 0 and 1 are identified is homeomorphic to the circle.

7. **[a]** Define what it means for a set $A$ to be a retract of $X$.

   **[b]** If $A$ is a retract of $X$ and both are pathwise connected, show that the homomorphism induced by the inclusion map from $A$ to $X$ is injective.

   **[c]** Define a deformation retract from $X$ to $A$ and state what it implies about the injective homomorphism from part **[b]**.

   **[d]** Use **[c]** to compute the fundamental group of the punctured plane $R^2 \backslash \{(0, 0)\}$. You can assume that the fundamental group of the circle is $Z$.

8. Letting $S^1$ be the circle, consider the mapping $f$ from $X = S^1 \times S^1$ to $Y = S^1 \times S^1$ given by $(a, b)$ goes to $(a^{-2}, b^4)$ where multiplication refers to complex multiplication thinking of $S^1$ as a subset of the complex numbers.

   **[a]** This a covering map. Consider the mapping $g$ from $[0, 1]$ to $Y$ given by $x \to (e^{6\pi ix}, e^{\pi ix})$. Find a lifting of $g$.

   **[b]** Describe precisely the induced group homomorphism $f^*$ on the fundamental groups, making sure to describe also the image of this group homomorphism. (You do not need to have done part a in order to be able to do this.)