Semi-direct products of groups by J. Steif, January, 2019

Again, there are tons of lecture notes on this topic on the web, for example, search for the very nice notes by Keith Conrad. (Let me know any corrections!).

Our goal is to apply the theorem from "Group actions part 2" to determine the fundamental group of the Klein bottle. This will be in the next set of notes. In these set of notes, we want to introduce a standard construction in group theory called the semi-direct product.

As said above, our goal in the next set of notes is to determine the fundamental group of the Klein bottle. But what does it mean to determine a group? Of course, we cannot write down a multiplication table when the group is infinite (and even if finite, a multiplication table would tend to give very little insight into the group). The best thing perhaps is to describe the group in terms of some standard constructions in group theory.

We will in the next set of notes, show that the fundamental group of the Klein bottle is a particular semi-direct product of Z with itself which will be described here.

The point of these notes is to describe, as briefly as possible, what a semi-direct product of groups is.

Background on direct product of groups

The hope is this subsection is known to the reader. It will be the simplest example of a semi-direct product of groups.

If N and H and groups, we can define the **(external!)** direct product of N and H, denoted $N \times H$, in the following trivial way. The elements are $N \times H$ and the multiplication is simply $(g_1, h_1)(g_2, h_2)$ is defined to be $(g_1 g_2, h_1 h_2)$. Verfy this is a group and determine the identity and the inverses of the elements.

The idea is that the construction is so simple that we feel that we understand $N \times H$ if we understand N and H separately.

How do we see **internally** if a group is isomorphic to a product as above? The answer is given by the notion of an internal direct product.

Definition. A group G is the **internal direct product** of its subgroups N and H if
(1) N and H are both normal subgroups of G
(2) G=NH (the normality of at least one of them guarantees that NH is a subgroup)
and (3) $N \cap H = e$.

The following is the key theorem.

Theorem. If G is the internal direct product of N and H, then G is isomorphic to $N \times H$.

Proof. Exercise. Hint. Map $N \times H$ to G by (n, h) is sent to nh. Using the properties of an internal direct product, one can show that this is an isomorphism. One key point to do this is that the elements of N and H must commute. Why is this the case? $nh = hn$ iff $nhn^{-1}h^{-1} = e$. This product however is in N since N is normal and it is in H since H is normal. Hence it is e. "QED"

Remark. Note that the external direct product $N \times H$ is an internal direct product of the subgroups $N \times \{e_H\}$ and $\{e_N\} \times H$.

Now semi-direct products of groups

First we do internal semi-direct products of groups

Definition. A group G is the internal semi-direct product of its subgroups N and H if
(1) N is a normal subgroup of G
(2) G=NH
and (3) $N \cap H = e$.

Note that the only difference between this and a direct product is that H need not be a normal subgroup.

Example. S_3 is an internal semi-direct product of the subgroups N and H where N is the subgroup generated by one of the two 3-cycles and H is the subgroup generated by any transposition. (N is normal having index 2.)

How does multiplication look in this situation? First the mapping from $N \times H$ to G given by (n, h) is sent to nh is a bijection **of sets**. It is onto by assumption and if $nh = n'h'$, then $n'^{-1}n = h'h^{-1}$, which means both sides are e being in both N and H. Hence $n' = n$ and $h' = h$, proving injective. Next, with the above identification, we have

$$(n, h)(n', h') = nhn'h' = nhn'h^{-1}hh' = (nhn'h^{-1}, hh').$$

So it is almost like a direct product. The second coordinates are multiplied like in a direct product. However, as far as the first coordinates, the second first coordinate, n', is first conjugated by h (leaving it in N since N is normal) and then it is multiplied it by n on the left.

External semi-direct products of groups

Just as we had both an external and internal way to look at a direct product, we also have that here. While the building blocks for an external direct product are just two groups, the building blocks for an external semi-direct product consist of two groups N and H and a homomorphism ϕ from H to $Aut(N)$ where the latter is the group of (group-)automorphisms of N.

Remark. The latter basically gives a group action of H on N with the additional condition that for each $h \in H$, the corresponding bijection on N is a group automorphism.

With the above building blocks, the external semi-direct product for N, H and ϕ is the group whose set is $N \times H$ with multiplication given by

$$(n, h)(n', h') = (n\phi_h(n'), hh')$$

where ϕ_h is the group automorphism of N corresponding to h. This group is denoted by $N \times_\phi H$.

Important exercise.
1. Verify this is a group, determining the identity and inverses.
2. Verify that $N \times_\phi H$ is the internal semi-direct product of the subgroups $N \times \{e_H\}$ and $\{e_N\} \times H$.
3. Show that the multiplication in a semi-direct product is the same as the product in a direct product if and only if ϕ is trivial, meaning ϕ_h is the identity in $Aut(N)$ for all h, meaning that for all h and n, $\phi_h(n) = n$.
4. Show that if ϕ is nontrivial, then $N \times_\phi H$ is never abelian.

The correspondence between internal and external semi-direct products, which is more subtle than for direct products, is the following.

Theorem. Assume G is the internal semi-direct product of N and H. (We always list the normal subgroup first and it is, not surprisingly, denoted by N.) Then G is isomorphic to $N \times_\phi H$ where $\phi_h(n) = hnh^{-1}$ (where the multiplication is defined in the original group G).

Exercise. Check that ϕ above gives a homomorphism from H to $Aut(N)$ and prove the result.

Remark. We think of a semi-direct product as a direct product "with a twist". We twist the second first coordinate using the first second coordinate before it is multiplied by the first first coordinate but otherwise it is like a direct product.

Philosophical point. We maybe can now say that we sort of understand semi-direct products and if someone describes a complicated group in this way, we might be satisfied with that description.

Do semi-direct products come up often? All the time, at least for small groups. If we stick to non-abelian groups of order at most 15 (abelian groups are characterized to just be direct products of cyclic groups), then every one of these is a semi-direct product of two cyclic groups with two exceptions. First, the alternating group A_4 on 12 elements is still a semi-direct product, but not of two cyclic groups, but rather of $Z/2 \times Z/2$ and $Z/3$ and secondly, the quaternions on 8 elements is not a semi-direct product of any two things (except of course itself and e).

Exercise. Consider $Z/n \times_\phi Z/2$ where ϕ_0 is the identity (as it has to be) and $\phi_1(x) = x^{-1}$. Verify this is a valid semi-direct product and convince yourself it is just the so-called Dihedral group which corresponds to the symmetry group of an n-gon.

Semi-direct products can be subtle. Here is a subtle fact. It can happen that we have a semi-direct product $N \times_\phi H$ with ϕ nontrivial but such that $N \times_\phi H$ is isomorphic to $N \times H$. (Since the products are different, it cannot be the case that the identity map is an isomorphism but there still might be some other map which gives an isomorphism). Note that this cannot happen if N and H are both abelian.

Lemma. There are exactly two external semi-direct products of Z with itself.
Proof. First note that $Aut(Z)$ is just $Z/2$ since there are only two group automorphisms of Z, given by x goes to x (the identity) or x goes to $-x$. Now we have to know how many homomorphisms there are from Z to $Z/2$. There are only two. Writing $Z/2$ as ± 1 with multiplication, either the map from Z to $Z/2$ sends every x in Z to 1, which corresponds to ϕ being trivial and hence we get a direct product or we have the map from Z to $Z/2$ which sends every x to $(-1)^x$. Why is that? The point is that since 1 generates Z, everything is determined by where 1 goes under ϕ. If it goes to 1, we have the first case, and if it does to -1, we are in the second case since once 1 is sent to -1, the homomorphism property of ϕ forces upon us that x goes to $(-1)^x$. QED

Verify that in equations, the second external semi-direct product becomes

$$(n, m)(n', m') = (n + (-1)^m n', mm').$$

Why did we spend so much time looking at this very specific external semi-direct product? The reason is that it turns out to be the fundamental group of the Klein bottle as we will see!