Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$ **,CTH**$[TMV100]$**)**
August 16, 2010, morning , V.
No aids.
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Each problem is worth 3 points.

1. Let $(X, \mathcal{M}, \mu)$ be a positive measure space, $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of positive real numbers. Set

$$\nu(A) = \sum_{k=1}^{n} c_k \mu(A \cap E_k),\ \ A \in \mathcal{M}.$$

Show that ν is absolutely continuous with respect to μ and find its Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution. If $A \in \mathcal{M}$ and $\mu(A) = 0$ we have $\mu(A \cap E_k) = 0$ for $k = 1, ..., n$ and it follows that $\nu(A) = 0$. Hence $\nu << \mu$. Moreover, if $A \in \mathcal{M}$,

$$\nu(A) = \sum_{k=1}^{n} c_k \int_A \chi_{E_k} d\mu = \int_A \sum_{k=1}^{n} c_k \chi_{E_k} d\mu$$

and thus

$$\frac{d\nu}{d\mu} = \sum_{k=1}^{n} c_k \chi_{E_k}.$$

2. Suppose $a > 1$. Show that

$$\int_0^\infty \frac{x^{a-1}}{e^{2x} - 1} dx = 2^{-a} \Gamma(a) \varsigma(a)$$

where

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

and

$$\varsigma(a)=\sum_{n=1}^{\infty}n^{-a}.$$

Solution. We have

$$I=_{def}\int_0^{\infty}\frac{x^{a-1}}{e^{2x}-1}dx=2^{-a}\int_0^{\infty}\frac{x^{a-1}e^{-x}}{1-e^{-x}}dx$$

$$=2^{-a}\int_0^{\infty}x^{a-1}e^{-x}\sum_{n=0}^{\infty}e^{-nx}dx.$$

Thus by monotone convergence

$$I=2^{-a}\sum_{n=0}^{\infty}\int_0^{\infty}x^{a-1}e^{-x}e^{-nx}dx$$

$$=2^{-a}\sum_{n=0}^{\infty}\frac{1}{(n+1)^a}\int_0^{\infty}y^{a-1}e^{-y}dy=2^{-a}\Gamma(a)\sum_{n=0}^{\infty}\frac{1}{(n+1)^a}$$

$$=2^{-a}\Gamma(a)\varsigma(a).$$

3. Suppose

$$\mu(A)=\mu_1(A)=\frac{1}{2}\int_A e^{-|t|}dt,\ \ A\in\mathcal{B}_{\mathbf{R}},$$

$\mu_2=\mu\times\mu$, ..., and $\mu_n=\mu_{n-1}\times\mu$, $n\geq 2$. Moreover, let $\varepsilon>0$ and define

$$A_n=\left\{x\in\mathbf{R}^n;\ \ ||\ x\ |^2-2n\ |\leq\varepsilon n\right\}$$

where $|\ x\ |=\sqrt{x_1^2+...+x_n^2}$ if $x=(x_1,...,x_n)\in\mathbf{R}^n$. Show that

$$\mu_n(A_n^c)\leq\frac{20}{n\varepsilon^2}$$

and conclude that

$$\lim_{n\to\infty}\mu_n(A_n)=1.$$

Solution. First note that μ_n is a probability measure and

$$\int_{\mathbf{R}}t^2d\mu(t)=2.$$

By the Markov inequality

$$\mu_n(A_n^c) \leq \frac{1}{n^2\varepsilon^2}\int_{\mathbf{R}^n}(\mid x\mid^2 - 2n)^2 d\mu_n(x)$$

$$= \frac{1}{n^2\varepsilon^2}\int_{\mathbf{R}^n}(\sum_1^n (x_k^2 - 2))^2 d\mu_n(x)$$

$$= \frac{1}{n^2\varepsilon^2}\sum_1^n \int_{\mathbf{R}^n}(x_k^2-2)^2 d\mu_n(x) + \frac{2}{n^2\varepsilon^2}\sum_{1\leq j<k\leq n}\int_{\mathbf{R}^n}(x_j^2-2)(x_k^2-2)d\mu_n(x).$$

Here, if $j \neq k$,

$$\int_{\mathbf{R}^n}(x_j^2-2)(x_k^2-2)d\mu_n(x) = \int_{\mathbf{R}}(x_j^2-2)d\mu(x_j)\int_{\mathbf{R}}(x_k^2-2)d\mu(x_k) = 0$$

and we get

$$\mu_n(A_n^c) \leq \frac{C}{n\varepsilon^2}$$

where

$$C = \int_{\mathbf{R}}(t^2-2)^2 d\mu(t) = 20.$$

Consequently,

$$\lim_{n\to\infty}\mu_n(A_n^c) = 0$$

and since $1 - \mu_n(A_n^c) = \mu_n(A_n) \leq 1$, we have

$$\lim_{n\to\infty}\mu_n(A_n) = 1.$$

4. Let $\mathcal{C}$ be a collection of open balls in $\mathbf{R}^n$ and let $V = \cup_{B\in\mathcal{C}}B$. Prove that to each $c < m_n(V)$, there exist pairwise disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\Sigma_{i=1}^k m_n(B_i) > 3^{-n}c.$$

(Here m_n denotes Lebesgue measure on $\mathbf{R}^n$.)

5. State and prove the Hahn decomposition theorem.