LÖSNINGAR
INTEGRATIONSTEORI (5p)
(GU$[MAF440]$,**CTH**$[TMV100]$**)**
Dag, tid: 8 oktober 2004 fm
Hjälpmedel: Inga.

1. Suppose

$$f_n(x) = n \mid x \mid e^{-\frac{nx^2}{2}},\ x \in \mathbf{R},\, n \in \mathbf{N}_+.$$

Show that there is no $g \in L^1(m)$ such that $f_n \leq g$ for all $n \in \mathbf{N}_+$.

Solution. We have

$$\lim_{n\to\infty} f_n(x) = 0,\ \text{all } x \in \mathbf{R}$$

and

$$\int_{-\infty}^{\infty} f_n(x)dx = \left[\sqrt{n}x = y\right] = \int_{-\infty}^{\infty} \mid y \mid e^{-\frac{y^2}{2}} dy\ = 2,\ \text{all } n \in \mathbf{N}_+.$$

The Lebesgue Dominated Convergence Theorem now implies that there is no $g \in L^1(m)$ such that $\mid f_n \mid \leq g$ for all $n \in \mathbf{N}_+$. Since $f_n = \mid f_n \mid$ we are done.

2. Set

$$f(x) = \lim_{T\to\infty} \int_0^T \frac{\sin t}{x+t} dt,\ x \geq 0$$

and

$$g(x) = \frac{f(x)}{\sqrt{x}},\ x \geq 0.$$

Prove that g is Lebesgue integrable on $[0, \infty[$.

Solution. Let $x \geq 0$. By partial integration

$$\int_{\frac{\pi}{2}}^{T} \frac{\sin t}{x+t} dt = -\frac{\cos T}{x+T} - \int_{\frac{\pi}{2}}^{T} \frac{\cos t}{(x+t)^2} dt$$

and we get

$$f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin t}{x+t} dt - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos t}{(x+t)^2} dt.$$

Note that f is a Borel function by the Tonelli Theorem. Moreover,

$$| f(x) | \leq \int_0^{\frac{\pi}{2}} \frac{| \sin t |}{t} dt + \int_{\frac{\pi}{2}}^{\infty} \frac{1}{(x+t)^2} dt$$

and since $| \sin t | \leq t$ for $t \geq 0$, we get

$$| f(x) | \leq \frac{\pi}{2} + \frac{1}{x + \frac{\pi}{2}} \leq \frac{\pi}{2} + \frac{2}{\pi}.$$

Hence

$$\int_0^1 \frac{| f(x) |}{\sqrt{x}} dx < \infty.$$

Furthermore,

$$\begin{aligned} | f(x) | &\leq \int_0^{\frac{\pi}{2}} \frac{1}{x} dt + \int_{\frac{\pi}{2}}^{\infty} \frac{1}{(x+t)^2} dt \\ &= \frac{\pi}{2x} + \frac{1}{x + \frac{\pi}{2}} \leq (\frac{\pi}{2} + 1)\frac{1}{x} \end{aligned}$$

and it follows that

$$\int_1^{\infty} \frac{| f(x) |}{\sqrt{x}} dx < \infty.$$

Summing up we conclude that g is Lebesgue integrable on $[0, \infty[$.

3. a) Let $\mathcal{M}$ be an algebra of subsets of X and $\mathcal{N}$ an algebra of subsets of Y. Furthermore, let S be the set of all finite unions of sets of the type $A \times B$, where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Prove that S is an algebra of subsets of $X \times Y$.

b) Assume $\mathcal{M}$ is a σ-algebra of subsets of X and $\mathcal{N}$ a σ-algebra of subsets of Y and let $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu)$ be a finite positive measure space. Prove that to each $E \in \mathcal{M} \otimes \mathcal{N}$ and $\varepsilon > 0$ there exists $F \in S$ such that

$$\mu(E \Delta F) < \varepsilon.$$

(Here S is as in Part a).)

Solution. a) The main point in the proof is to show that S is closed under finite intersections. To see this let

$$E = \cup_{k=1}^{M} (A_k \times B_k)$$

and

$$F = \cup_{k=1}^{N} (C_k \times D_k)$$

where $A_1, ..., A_M,\ C_1, ..., C_N \in \mathcal{M}$ and $B_1, ..., B_M,\ D_1, ..., D_N \in \mathcal{N}$. It is enough to prove that $E \cap F \in S$. But

$$E \cap F = \cup_{\substack{1 \le i \le M \\ 1 \le j \le N}} ((A_i \cap C_j) \times (B_i \cap D_j))$$

and we are done.

To prove that S is an algebra first note that $\phi \in S$ and that S is closed under finite unions. If E is as above it remains to prove that the complement E^c belongs to S. But

$$E^c = \cap_{k=1}^{M} (A_k \times B_k)^c$$
$$= \cap_{k=1}^{M} ((A_k^c \times Y) \cup (X \times B_k^c))$$

and it follows $E^c \in S$.

b) Let Σ be the class of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the property in b) holds. Clearly, $\phi \in \Sigma$. Now let $E \in \Sigma$. If $F \in S$, then $F^c \in S$ and $E \Delta F = E^c \Delta F^c$. Hence $E^c \in \Sigma$.

Finally, let $E_i \in \Sigma$, $i \in \mathbf{N}_+$. We shall prove that $E = \cup_{i=1}^{\infty} E_i \in \Sigma$. To this end let $\varepsilon > 0$ be arbitrary and choose $F_i \in S$ such that

$$\mu(E_i \Delta F_i) < 2^{-i} \varepsilon$$

for all $i \in \mathbf{N}_+$. Since

$$E \Delta (\cup_{i=1}^{\infty} F_i) \subseteq \cup_{i=1}^{\infty} E_i \Delta F_i,$$

$$\mu(E \Delta (\cup_{i=1}^{\infty} F_i) \le \Sigma_{i=1}^{\infty} \mu(E_i \Delta F_i) < \varepsilon.$$

Now

$$E \Delta (\cup_{i=1}^{\infty} F_i) = (\cap_{i=1}^{\infty} (E \cap F_i^c)) \cup (E^c \cap (\cup_{i=1}^{\infty} F_i))$$

and since μ is a finite positive measure it follows that

$$\mu((\cap_{i=1}^{n} (E \cap F_i^c)) \cup (E^c \cap (\cup_{i=1}^{\infty} F_i))) < \varepsilon$$

if n is sufficiently large. Hence

$$\mu(E\Delta(\cup_{i=1}^{n}F_i)) = \mu((\cap_{i=1}^{n}(E\cap F_i^c))\cup(E^c\cap(\cup_{i=1}^{n}F_i))) < \varepsilon$$

if n is large, which proves that $\cup_{i=1}^{\infty}E_i \in \Sigma$. Thus Σ is a σ-algebra contained in $\mathcal{M}\otimes\mathcal{N}$ and since Σ contains all measurable rectangles $\Sigma = \mathcal{M}\otimes\mathcal{N}$.

4. Formulate and prove the Fatou's Lemma.

5. Let $\mathcal{C}$ be a collection of open balls in $\mathbf{R}^n$ and set $V = \cup_{B\in\mathcal{C}}B$. Prove that to each $c < m_n(V)$ there exist pairwise disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\Sigma_{i=1}^{k} m_n(B_i) > 3^{-n}c.$$

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INTEGRATIONSTEORI (5p)
(GU$[MAF440]$ **,CTH**$[TMV100]$**)**
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1. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $(Y, \mathcal{N})$ a measurable space. Furthermore, suppose $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and let $\nu = \mu f^{-1}$, that is $\nu(B) = \mu(f^{-1}(B))$ if $B \in \mathcal{N}$. Show that f is $(\mathcal{M}^-, \mathcal{N}^-)$-measurable, where $\mathcal{M}^-$ denotes the completion of $\mathcal{M}$ with respect to μ and $\mathcal{N}^-$ the completion of $\mathcal{N}$ with respect to ν.

Solution: Suppose $B \in \mathcal{N}^-$. We will prove that $f^{-1}(B) \in \mathcal{M}^-$. To this end, choose $B_0, B_1 \in \mathcal{N}$ such that $B_0 \subseteq B \subseteq B_1$ and $\nu(B_1 \setminus B_0) = 0$. Then

$f^{-1}(B_0)$, $f^{-1}(B_1) \in \mathcal{M}$ and $f^{-1}(B_0) \subseteq f^{-1}(B) \subseteq f^{-1}(B_1)$. Furthermore, $f^{-1}(B_1) \setminus f^{-1}(B_0) = f^{-1}(B_1 \setminus B_0)$ and we get

$$\mu(f^{-1}(B_1) \setminus f^{-1}(B_0)) = \nu(B_1 \setminus B_0) = 0.$$

Thus $f^{-1}(B) \in \mathcal{M}^-$ and we are done.

2. Compute the following limit and justify the calculations:

$$\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^{n^2} e^{-nx} dx.$$

Solution. We have

$$\begin{aligned}(1+\frac{x}{n})^{n^2} e^{-nx} &= e^{n^2 \ln(1+\frac{x}{n})-nx} \\ &= e^{n^2(\frac{x}{n}-\frac{x^2}{2n^2}+(\frac{x}{n})^3 B(\frac{x}{n}))-nx}\end{aligned}$$

where B is bounded in a neighbourhood of the origin. Accordingly from this,

$$\lim_{n\to\infty} (1+\frac{x}{n})^{n^2} e^{-nx} = e^{-\frac{x^2}{2}}.$$

To find a majorant, let $x \geq 0$ be fixed and introduce the function

$$f(n) = n^2 \ln(1+\frac{x}{n}) - nx$$

defined for all real $n \geq 1$. We claim that

$$f'(n) = 2n \ln(1+\frac{x}{n}) - \frac{2x+\frac{x^2}{n}}{1+\frac{x}{n}} \leq 0.$$

To see this put

$$g(t) = 2(1+t)\ln(1+t) - (2t+t^2) \text{ for } t \geq 0$$

and note that $f'(n) \leq 0$ if and only if $g(\frac{x}{n}) \leq 0$. But $g(0) = 0$ and

$$g'(t) = 2(\ln(1+t) - t) \leq 0$$

and it follows that $g \leq 0$. Thus $f'(n) \leq 0$ and, hence, $f(n) \leq f(1)$. Now

$$(1+\frac{x}{n})^{n^2}e^{-nx} \leq (1+x)e^{-x} \in L^1(m_{0,\infty})$$

where $m_{0,\infty}$ is Lebesgue measure on $[0,\infty[$ and the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to\infty}\int_0^\infty (1+\frac{x}{n})^{n^2}e^{-nx}dx = \int_0^\infty e^{-\frac{x^2}{2}}dx = \sqrt{\frac{\pi}{2}}.$$

3. Suppose $a > 0$ and

$$\mu_a = e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\delta_n$$

where δ_n is the Dirac measure on $\mathbf{N} = \{0,1,2,...\}$ at the point $n \in \mathbf{N}$, that is $\delta_n(A) = \chi_A(n)$ if $n \in \mathbf{N}$ and $A \subseteq \mathbf{N}$. Prove that

$$(\mu_a \times \mu_b)s^{-1} = \mu_{a+b}$$

for all $a, b > 0$ if $s(x,y) = x+y$, $x, y \in \mathbf{N}$.

Solution. If μ and ν are finite positive measures on $\mathbf{N}$, we define $\mu * \nu = (\mu \times \nu)s^{-1}$. Now, given $a, b > 0$ and $A \in \mathcal{P}(\mathbf{N})$, the Tonelli Theorem implies that

$$\begin{aligned}(\mu_a * \mu_b)(A) &= (\mu_a \times \mu_b)(\{(x,y) \in \mathbf{N}^2;\ x+y \in A\}) \\ &= \int_{\mathbf{N}} \mu_a(\{x \in \mathbf{N};\ x+y \in A\})d\mu_b(y)\end{aligned}$$

and by applying the Lebesgue Monotone Convergence Theorem we have,

$$(\mu_a * \mu_b)(A) = \sum_{i=0}^{\infty} e^{-a}\frac{a^i}{i!}\int_{\mathbf{N}}\delta_i(\{x \in \mathbf{N};\ x+y \in A\})d\mu_b(y)$$

$$= \sum_{i=0}^{\infty} e^{-a}\frac{a^i}{i!}(\delta_i * \mu_b)(A).$$

In a similar way,

$$(\delta_i * \mu_b)(A) = \sum_{j=0}^{\infty} e^{-b} \frac{b^j}{j!} (\delta_i * \delta_j)(A).$$

Since $\delta_i * \delta_j = \delta_{i+j}$, we get

$$(\mu_a * \mu_b)(A) = \sum_{i,j=0}^{\infty} e^{-(a+b)} \frac{a^i b^j}{i!j!} \delta_{i+j}(A)$$

$$= \sum_{n=0}^{\infty} (e^{-(a+b)} \delta_n(A) \sum_{\substack{i+j=n \\ i.j \geq 0}} \frac{a^i b^j}{i!j!}) = \sum_{n=0}^{\infty} e^{-(a+b)} \frac{(a+b)^n}{n!} \delta_n(A) = \mu_{a+b}(A).$$

4. Suppose $f :]a,b[\times X \to \mathbf{R}$ is a function such that $f(t, \cdot) \in \mathcal{L}^1(\mu)$ for each $t \in]a,b[$ and, moreover, assume $\frac{\partial f}{\partial t}$ exists and

$$\mid \frac{\partial f}{\partial t}(t,x) \mid \leq g(x) \text{ for all } (t,x) \in]a,b[\times X$$

where $g \in \mathcal{L}^1(\mu)$. Set

$$F(t) = \int_X f(t,x) d\mu(x) \text{ if } t \in]a,b[\,.$$

Prove that F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(t,x) d\mu(x) \text{ if } t \in]a,b[\,.$$

5. Suppose θ is an outer measure on X and let $\mathcal{M}(\theta)$ be the set of all $A \subseteq X$ such that

$$\theta(E) = \theta(E \cap A) + \theta(E \cap A^c) \text{ for all } E \subseteq X.$$

Prove that $\mathcal{M}(\theta)$ is a σ-algebra and that the restriction of θ to $\mathcal{M}(\theta)$ is a complete measure.

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INTEGRATIONSTEORI (5p)
(GU$[MAF440]$ **,CTH**$[TMV100]$**)**
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1. Suppose $f(x) = x\cos(\pi/x)$ if $0 < x < 2$ and $f(x) = 0$ if $x \in \mathbf{R}\setminus]0,2[$. Prove that f is not of bounded variation on $\mathbf{R}$.

Solution. We have

$$\Sigma_{k=1}^{n} \mid f(\frac{1}{k+1}) - f(\frac{1}{k}) \mid = \Sigma_{k=1}^{n} \mid \frac{1}{k+1}\cos(k+1)\pi - \frac{1}{k}\cos k\pi \mid$$

$$= \Sigma_{k=1}^{n}(\frac{1}{k+1} + \frac{1}{k}) = \frac{1}{n+1} + 1 + 2\Sigma_{k=2}^{n}\frac{1}{k} \to \infty \text{ as } n \to \infty.$$

2. Let $(X, \mathcal{M}, \mu)$ be a finite positive measure space and suppose $\varphi(t) = \min(t,1)$, $t \geq 0$. Prove that $f_n \to f$ in measure if and only if $\varphi(\mid f_n - f \mid) \to 0$ in $L^1(\mu)$.

Solution: $\Rightarrow$: For any $\varepsilon > 0$,

$$\int_X \varphi(\mid f_n - f \mid)d\mu \leq \int_{\mid f_n - f\mid \leq \varepsilon} \varphi(\mid f_n - f \mid)d\mu$$

$$+ \int_{\mid f_n - f\mid > \varepsilon} \varphi(\mid f_n - f \mid)d\mu \leq \int_{\mid f_n - f\mid \leq \varepsilon} \varphi(\varepsilon)d\mu + \int_{\mid f_n - f\mid > \varepsilon} 1 d\mu$$

$$\leq \varphi(\varepsilon)\mu(X) + \mu(\mid f_n - f \mid > \varepsilon).$$

Thus

$$0 \leq \limsup_{n\to\infty} \int_X \varphi(\mid f_n - f \mid)d\mu \leq \varphi(\varepsilon)\mu(X)$$

and by letting $\varepsilon \downarrow 0$,

$$\lim_{n\to\infty} \int_X \varphi(\mid f_n - f \mid)d\mu = 0.$$

$\Leftarrow$: For any $\varepsilon > 0$,

$$\mu(\mid f_n - f \mid > \varepsilon) \leq \mu(\varphi(\mid f_n - f \mid) \geq \varphi(\varepsilon))$$

and the Markov inequality gives

$$\mu(\mid f_n - f \mid > \varepsilon) \leq \frac{1}{\varphi(\varepsilon)} \int_X \varphi(\mid f_n - f \mid) d\mu.$$

Thus $\mu(\mid f_n - f \mid > \varepsilon) \to 0$ as $n \to \infty$.

3. Let P denote the class of all Borel probability measures on $[0,1]$ and L the class of all functions $f : [0,1] \to [-1,1]$ such that

$$\mid f(x) - f(y) \mid \leq \mid x - y \mid, \ x, y \in [0,1].$$

For any $\mu, \nu \in P$, define

$$\rho(\mu, \nu) = \sup_{f \in L} \mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \mid .$$

(a) Show that (P, ρ) is a metric space. (b) Compute $\rho(\mu, \nu)$ if μ is linear measure on $[0,1]$ and $\nu = \frac{1}{n}\Sigma_{k=0}^{n-1}\delta_{\frac{k}{n}}$, where $n \in \mathbf{N}_+$ (linear measure on $[0,1]$ is Lebesgue measure on $[0,1]$ restricted to the Borel sets in $[0,1]$).

Solution. (a): (1) Clearly, $\rho(\mu, \nu) \geq 0$ and

$$\rho(\mu, \nu) \leq \mu([0,1]) + \nu([0,1]) = 2 < \infty.$$

Moreover, if $\mu \neq \nu$ there is a compact set $K \subseteq [0,1]$ such that $\mu(K) \neq \nu(K)$. If $f_n(x) = \max(0, 1 - nd(x, K))$, $x \in [0,1]$, then $f_n \downarrow \chi_K$, and the Lebesgue Dominated Convergence Theorem implies that

$$\int_{[0,1]} f_n d\mu \neq \int_{[0,1]} f_n d\nu$$

if n is sufficiently large. But $\frac{1}{n} f_n \in L$, and, hence, if n is large

$$\rho(\mu, \nu) \geq \mid \int_{[0,1]} \frac{1}{n} f_n d\mu - \int_{[0,1]} \frac{1}{n} f_n d\nu \mid$$

$$= \frac{1}{n} \mid \int_{[0,1]} f_n d\mu - \int_{[0,1]} f_n d\nu \mid > 0.$$

Thus $\rho(\mu, \nu) > 0$.

(2) Since $\mid t \mid$ is an even function of t, $\rho(\mu, \nu) = \rho(\nu, \mu)$.

(3) If $f \in L$ and $\mu, \nu, \tau \in P$,

$$\mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \mid$$

$$\leq \mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\tau \mid + \mid \int_{[0,1]} f d\tau - \int_{[0,1]} f d\nu \mid$$

$$\leq \rho(\mu, \tau) + \rho(\tau, \nu)$$

and we get $\rho(\mu, \nu) \leq \rho(\mu, \tau) + \rho(\tau, \nu)$.

(b) If $f \in L$,

$$\mid \int_{[0,1]} f d\mu - \int_{[0,1]} f d\nu \mid = \mid \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=0}^{n-1} f(\frac{k}{n}) \mid$$

$$= \mid \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (f(x) - f(\frac{k}{n})) dx \mid$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mid f(x) - f(\frac{k}{n}) \mid dx$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mid x - \frac{k}{n} \mid dx = \frac{1}{2n}$$

where equality occurs if $f(x) = x$. Thus $\rho(\mu, \nu) = \frac{1}{2n}$.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $w : X \to [0, \infty]$ a measurable function. Define

$$\nu(A) = \int_A w d\mu, \ A \in \mathcal{M}.$$

Prove that ν is a positive measure and

$$\int_X f d\nu = \int_X f w d\mu$$

for every measurable function $f : X \to [0, \infty]$.

5. Suppose $f \in L^1_{loc}(m_n)$ and set

$$(A_r f)(x) = \frac{1}{m_n(B(x,r))} \int_{B(x,r)} f(y)dy,\ (x,r) \in \mathbf{R}^n \times]0, \infty[$$

where $B(x,r)$ is the open ball of centre $x \in \mathbf{R}^n$ and radius $r > 0$ (with respect to the Euclidean metric $d(x,y) =\mid x - y \mid$).

(a) Set

$$f^*(x) = \sup_{r>0} \mid (A_r f)(x) \mid,\ x \in \mathbf{R}^n.$$

Prove that

$$\{f^* \geq \lambda\} \in \mathcal{B}(\mathbf{R}^n) \text{ if } \lambda \geq 0.$$

(b) Use the (Wiener) Maximal Theorem to prove that

$$\lim_{r \to 0+} (A_r f)(x) = f(x) \text{ a.e. } [m_n].$$

LÖSNINGAR

INTEGRATIONSTEORI (5p)

(GU$[MAF440]$,**CTH**$[TMV100]$**)**

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Hjälpmedel: Inga.

Skrivtid: 5 timmar

1. Suppose

$$f(t) = \int_0^\infty \frac{xe^{-x^2}}{x^2 + t^2} dx,\ t > 0.$$

Compute

$$\lim_{t\to 0+} f(t) \text{ and } \int_0^\infty f(t)dt.$$

Finally, prove that f is differentiable.

Solution. Suppose $t_n \downarrow 0$. Then for each $x > 0$, $\frac{xe^{-x^2}}{x^2+t_n^2} \uparrow \frac{1}{x}e^{-x^2}$ and the LMCT implies that

$$\int_0^\infty \frac{xe^{-x^2}}{x^2+t_n^2}dx \uparrow \int_0^\infty \frac{1}{x}e^{-x^2}dx = \infty$$

since $e^{-x^2} > \frac{1}{3}\chi_{[0,1]}(x)$ if $x \geq 0$ and

$$\int_0^1 \frac{1}{x}dx = \infty.$$

Hence

$$\lim_{t\to 0+} f(t) = \infty.$$

Furthermore, the Tonelli Theorem yields

$$\int_0^\infty f(t)dt = \int_0^\infty \left\{ \int_0^\infty \frac{xe^{-x^2}}{x^2+t^2}dt \right\} dx$$

$$= \int_0^\infty \left[e^{-x^2} \arctan \frac{t}{x} \right]_{t=0}^{t=\infty} dx = \frac{\pi}{2}\int_0^\infty e^{-x^2}dx = \frac{\pi^{\frac{3}{2}}}{4}.$$

Finally, it is enough to prove that $f(t)$ is differentiable on the interior of any given compact subinterval $[a, b]$ of $]0, \infty[$. To this end, first note that

$$\frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2} = -\frac{2txe^{-x^2}}{(x^2+t^2)^2}$$

and

$$\sup_{a\leq t\leq b} \mid \frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2} \mid\leq \frac{2bxe^{-x^2}}{(x^2+a^2)^2} \in L^1(m_{0,\infty}).$$

Therefore, by a familiar result (Folland Theorem 2.27 or LN, Example 2.2.1) $f'(t)$ exists for all $a < t < b$ and equals

$$\int_0^\infty \frac{\partial}{\partial t}\frac{xe^{-x^2}}{x^2+t^2}dx = -2t\int_0^\infty \frac{xe^{-x^2}}{(x^2+t^2)^2}dx.$$

2. Suppose μ is a finite positive Borel measure on $\mathbf{R}^n$. (a) Let $(V_i)_{i\in I}$ be a family of open subsets of $\mathbf{R}^n$ and $V = \cup_{i\in I} V_i$. Prove that

$$\mu(V) = \sup_{\substack{i_1,...,i_k \in I \\ k\in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}).$$

(b) Let $(F_i)_{i\in I}$ be a family of closed subsets of $\mathbf{R}^n$ and $F = \cap_{i\in I} F_i$. Prove that

$$\mu(F) = \inf_{\substack{i_1,...,i_k \in I \\ k\in \mathbf{N}_+}} \mu(F_{i_1} \cap ... \cap F_{i_k}).$$

Solution. (a) Since $V \supseteq V_{i_1} \cup ... \cup V_{i_k}$ for all $i_1, ...i_k \in I$ and $k \in \mathbf{N}_+$,

$$\mu(V) \geq \sup_{\substack{i_1,...,i_k \in I \\ k\in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}).$$

To prove the reverse inequality first note that

$$\mu(A) = \sup_{\substack{K\subseteq A \\ K \text{ compact}}} \mu(K)$$

if $A \in \mathcal{R}_n$. Now first choose $\varepsilon > 0$ and then a compact subset K of $\mathbf{R}^n$ such that

$$\mu(K) > \mu(V) - \varepsilon.$$

Then there are finitely many $i_1, ..., i_k \in I$ such that $V_{i_1} \cup ... \cup V_{i_k} \supseteq K$. Accordingly from this,

$$\mu(V_{i_1} \cup ... \cup V_{i_k}) > \mu(V) - \varepsilon$$

and we get

$$\sup_{\substack{i_1,...,i_k \in I \\ k\in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}) \geq \mu(V).$$

Thus

$$\mu(V) = \sup_{\substack{i_1,...,i_k \in I \\ k\in \mathbf{N}_+}} \mu(V_{i_1} \cup ... \cup V_{i_k}).$$

b) Since μ is a finite measure, by Part (a)

$$\mu(F) = \mu(\mathbf{R}^n) - \mu(F^c)$$

$$= \mu(\mathbf{R}^n) - \sup_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(F_{i_1}^c \cup ... \cup F_{i_k}^c)$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} (\mu(\mathbf{R}^n) - \mu(F_{i_1}^c \cup ... \cup F_{i_k}^c))$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu((F_{i_1}^c \cup ... \cup F_{i_k}^c)^c)$$

$$= \inf_{\substack{i_1,...,i_k \in I \\ k \in \mathbf{N}_+}} \mu(F_{i_1} \cap ... \cap F_{i_k}).$$

3. Suppose f and g are real-valued absolutely continuous functions on the compact interval $[a, b]$. Show that the function $h = \max(f, g)$ is absolutely continuous and $h' \leq \max(f', g')$ a.e. $[m_{a,b}]$ ($m_{a,b}$ denotes Lebesgue measure on $[a, b]$).

Solution. If $(A_i)_{1 \leq i \leq 2}$ and $(B_i)_{1 \leq i \leq 2}$ are sequences of real numbers

$$A_i \leq B_i + \mid A_i - B_i \mid$$

$$\leq B_i + \max_{1 \leq i \leq 2} \mid A_i - B_i \mid \leq \max_{1 \leq i \leq 2} B_i + \max_{1 \leq i \leq 2} \mid A_i - B_i \mid$$

and, hence,

$$\max_{1 \leq i \leq 2} A_i \leq \max_{1 \leq i \leq 2} \mid A_i - B_i \mid + \max_{1 \leq i \leq 2} B_i$$

and

$$\max_{1 \leq i \leq 2} A_i - \max_{1 \leq i \leq 2} B_i \leq \max_{1 \leq i \leq 2} \mid A_i - B_i \mid .$$

Thus, by interchanging A_i and B_i,

$$\mid \max_{1 \leq i \leq 2} A_i - \max_{1 \leq i \leq 2} B_i \mid \leq \max_{1 \leq i \leq 2} \mid A_i - B_i \mid .$$

Next choose $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\Sigma_{k=1}^n \mid f(a_k) - f(b_k) \mid < \varepsilon/2$$

and

$$\Sigma_{k=1}^{n} \mid g(a_k) - g(b_k) \mid < \varepsilon/2$$

if $n \in \mathbf{N}_+$ and $]a_i, b_i[\,$, $i = 1, ..., n$, are mutually disjoint subintervals of $[a, b]$. Thus, for such intervals

$$\Sigma_{k=1}^{n} \mid h(a_k) - h(b_k) \mid$$

$$\leq \Sigma_{k=1}^{n} \max(\mid f(a_k) - f(b_k) \mid, \mid g(a_k) - g(b_k) \mid)$$
$$\leq \Sigma_{k=1}^{n} (\mid f(a_k) - f(b_k) \mid + \mid g(a_k) - g(b_k) \mid) < \varepsilon$$

and it follows that h is absolutely continuous.

As above it follows that

$$\max_{1 \leq i \leq 2} A_i - \max_{1 \leq i \leq 2} B_i \leq \max_{1 \leq i \leq 2} (A_i - B_i).$$

Therefore, for each $x \in\,]a, b[$ and $\omega \in\,]0, b - x[\,$,

$$h(x + \omega) - h(x) \leq \max(f(x + \omega) - f(x), g(x + \omega) - g(x))$$

and

$$\frac{h(x + \omega) - h(x)}{\omega} \leq \max(\frac{f(x + \omega) - f(x)}{\omega}, \frac{g(x + \omega) - g(x)}{\omega}).$$

Since f, g, and h are absolutely continuous, by letting $\omega \downarrow 0$, we get $h'(x) \leq \max(f'(x), g'(x))$ for $m_{a,b}$-almost all $x \in [a, b]$.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. (a) If $f_n \to f$ in measure and $f_n \to g$ in measure, show that $f = g$ a.e. $[\mu]$. (b) If $f_n \to f$ in L^1, show that $f_n \to f$ in measure.

5. (Lebesgue's Dominated Convergence Theorem) Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, measurable functions such that

$$\mid f_n(x) \mid \leq g(x), \text{ all } x \in X \text{ and } n \in \mathbf{N}_+$$

where $g \in \mathcal{L}^1(\mu)$. Moreover, suppose the limit $\lim_{n\to\infty} f_n(x)$ exists and equals $f(x)$ for every $x \in X$.

Prove that $f \in \mathcal{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

LÖSNINGAR
INTEGRATIONSTEORI (5p)
(GU[$MAF440$] ,**CTH**[$TMV100$]**)**
Dag, tid, sal: 9 sept 2006, fm, v
Hjälpmedel: Inga
Skrivtid: 5 timmar

1. Suppose $f \in L^1(m)$ and $\int_{\mathbf{R}} \mid f \mid dm > 0$, where m is Lebesgue measure on $\mathbf{R}$. Moreover, define

$$g(x) = \sup_{I \in \mathcal{I}_x} \frac{1}{m(I)} \int_I \mid f \mid dm$$

where for each $x \in \mathbf{R}$, $\mathcal{I}_x$ denotes the class of all open, non-empty intervals I such that $x \in I$. Prove that the level set $\{g > c\}$ is open for each real c and that $g \notin L^1(m)$.

Solution. If $x \in \{g > c\}$ there is an $I \in \mathcal{I}_x$ such that

$$\frac{1}{m(I)} \int_I \mid f \mid dm > c.$$

Hence $I \subseteq \{g > c\}$ and it follows that the set $\{g > c\}$ is open.

By the LDCT we find $a, b \in \mathbf{R}$ such that $a < b$ and

$$C =_{def} \int_a^b \mid f \mid dm > 0.$$

Now if $x \geq b$,

$$g(x) \geq \frac{1}{x+1-a}\int_a^{x+1} \mid f \mid dm$$
$$\geq \frac{C}{x+1-a}$$

and we conclude that

$$\int_b^\infty g(x)dx = \infty.$$

Consequently, $\int_{\mathbf{R}} gdm = \infty$ and $g \notin L^1(m)$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be an even Lebesgue measurable function such that $\int_0^\infty \mid f(x) \mid dx < \infty$ and define

$$g(x) = \int_{|x|}^\infty \frac{f(y)}{y}dy, \ x \neq 0$$

and

$$h(t) = \int_{-\infty}^\infty f(x)\cos txdx, \ t \in \mathbf{R}.$$

(a) Show that

$$\int_{-\infty}^\infty \mid g(x) \mid dx \leq \int_{-\infty}^\infty \mid f(x) \mid dx.$$

(b) Show that

$$\int_{-\infty}^\infty g(x)\cos txdx = \frac{1}{t}\int_0^t h(s)ds, \ t \neq 0.$$

(Hint for (b): First consider the case when f is an even and continuous function that vanishes in a neighbourhood of the origin and outside a bounded interval.)

Solution. (a) We have

$$\mid g(x) \mid \leq \int_{|x|}^\infty \frac{\mid f(y) \mid}{y}dy, \ x \neq 0.$$

Moreover, by the Tonelli Theorem

$$\int_{-\infty}^{\infty} \mid g(x) \mid dx \leq \iint\limits_{|x|\leq y} \frac{\mid f(y) \mid}{y} dxdy$$

$$= \int_{0}^{\infty} \frac{\mid f(y) \mid}{y} \left(\int_{|x|\leq y} dx \right) dy = 2\int_{0}^{\infty} \mid f(y) \mid dy = \int_{-\infty}^{\infty} \mid f(y) \mid dy.$$

(b) The space of all real-valued continuous functions with compact support is dense in $L^1(m_{0,\infty})$. Therefore by Part (a) and the LDCT it can be assumed that f is an even continuous function that vanishes in a neighbourhood of the origin and outside a bounded interval. In addition, since g and h are even we may assume $t > 0$. Now the function $f(x)\cos sx$, $x \in \mathbf{R}$, $0 \leq s \leq t$, is Lebesgue integrable and by integrating the relation

$$h(s) = \int_{-\infty}^{\infty} f(x) \cos sx dx$$

with respect to s over the interval $[0, t]$ the Fubini Theorem implies that

$$\int_{0}^{t} h(s)ds = \int_{-\infty}^{\infty} f(x) \left(\int_{0}^{t} \cos sx ds \right) dx = \int_{-\infty}^{\infty} f(x) \frac{\sin tx}{x} dx$$

$$= 2\int_{0}^{\infty} \frac{f(x)}{x} \sin tx dx = 2\left\{ [-g(x) \sin tx]_{0}^{\infty} + t\int_{0}^{\infty} g(x) \cos tx dx \right\}$$

$$= 2t\int_{0}^{\infty} g(x) \cos tx dx = t\int_{-\infty}^{\infty} g(x) \cos tx dx.$$

This proves Part (b).

3. Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and $f \in L^1(\mu)$. Define

$$g(t) = \int_{X} \mid f(x) - t \mid d\mu(x), \ t \in \mathbf{R}.$$

(a) Prove that

$$g(t) = g(a) + \int_{a}^{t} (\mu(f \leq s) - \mu(f \geq s))ds \ \text{ if } a, t \in \mathbf{R}.$$

(b) If $c \in \mathbf{R}$ and $\min g = g(c)$ show that

$$\mu(f \leq c) \geq \frac{1}{2}\mu(X)$$

and

$$\mu(f \geq c) \geq \frac{1}{2}\mu(X).$$

Solution. The special case $\mu = 0$ is trivial and it is enough to consider the case when $\mu(X) > 0$. Moreover by replacing μ by $\mu/\mu(X)$ and g by $g/\mu(X)$ it may be assumed that μ is a probability measure.

(a) Suppose $\varepsilon > 0$ is given and let $]a_k, b_k[\,$, $k = 1, ..., n$, be disjoint open intervals such that $\Sigma_1^n (b_k - a_k) < \varepsilon$. Then

$$| g(a_k) - g(b_k) | = | \int_X | f(x) - a_k | - | f(y) - b_k | \, d\mu(x) |$$

$$\leq \int_X || f(x) - a_k | - | f(x) - b_k || \, d\mu(y)$$

$$\leq \int_X | (f(x) - a_k) - (f(x) - b_k) | \, d\mu(y) = | a_k - b_k |$$

and consequently

$$\sum_1^n | g(a_k) - g(b_k) | \leq \varepsilon.$$

This proves that g is absolutely continuous and therefore g' exists a.e. with respect to Lebesgue measure on $\mathbf{R}$ and

$$g(t) = g(a) + \int_a^t g'(s) ds.$$

Let $A = \{t \in \mathbf{R};\ \mu(f = t) > 0\}$ and note that A is at most denumerable. To compute $g'(s)$ for fixed $s \notin A$, let $(h_n)_0^\infty$ be a sequence of non-zero real numbers which converges to zero. Then

$$\frac{g(s + h_n) - g(s)}{h_n} = \int_X \frac{| s + h_n - f(x) | - | s - f(x) |}{h_n} d\mu(x)$$

$$= \int_{\{f \neq s\}} \frac{| s + h_n - f(x) | - | s - f(x) |}{h_n} d\mu(x).$$

Here

$$| \frac{| s + h_n - f(x) | - | s - f(x) |}{h_n} | \leq 1$$

and

$$\lim_{n \to \infty} \frac{| s + h_n - f(x) | - | s - f(x) |}{h_n} = \begin{cases} 1 \text{ if } s > f(x) \\ -1 \text{ if } s < f(x). \end{cases}$$

Thus the LDCT gives

$$g'(s) = \int_{\{f \neq s\}} (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu = \int_X (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu$$

$$= \mu(f < s) - \mu(f > s) = \mu(f \leq s) - \mu(f \geq s).$$

In particular,

$$g'(s) = \mu(f \leq s) - \mu(f \geq s)$$

a.e. with respect to Lebesgue measure on **R**, which proves Part (a).

(b) Since

$$g(t) \geq | t | - \int_X | f(x) | d\mu(x)$$

the continuous function g attains a minimum at a certain point c. Now

$$g(t) - g(c) = \int_c^t (\mu(f \leq s) + \mu(f < s) - 1) ds$$

and it follows that

$$\int_c^t (2\mu(f \leq s) - 1) ds \geq 0 \text{ if } t \geq c.$$

Note that the function $\mu(f \leq t)$ is a right continuous function of t. Therefore, if $2\mu(f \leq c) - 1 < 0$ then $2\mu(f \leq s) - 1 < 0$ for all $s > c$ sufficiently close to c which is a contradiction. Thus $\mu(f \leq c) \geq 1/2$. By replacing f by $-f$ and c by $-c$ it follows that $\mu(-f \leq -c) \geq 1/2$, that is $\mu(f \geq c) \geq 1/2$.

4. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and suppose $x \in X$ and $E \in \mathcal{M} \otimes \mathcal{N}$. Prove that the set $E_x = \{y \in Y;\ (x, y) \in E\}$ belongs to the σ-algebra $\mathcal{N}$.

5. (Egoroff's Theorem) Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and let f_n, $n \in \mathbf{N}_+$, and f be real-valued measurable functions on X such that $f_n \to f$ a.e. $[\mu]$ as $n \to \infty$. Show that to every $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Solutions:

INTEGRATION THEORY (7.5 hp)
(GU$[MMA110GU]$ **,CTH**$[TMV100]$**)**
October 22, 2009, morning (5 hours), H
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. Let $n \in \mathbf{N}_+$ and define $f_n(x) = e^x(1 - \frac{x^2}{2n})^n$, $x \in \mathbf{R}$. Compute

$$\lim_{n \to \infty} \int_{-\sqrt{2n}}^{\sqrt{2n}} f_n(x)dx.$$

Solution. We have

$$I_n =_{def} \int_{-\sqrt{2n}}^{\sqrt{2n}} f_n(x)dx = \int_{-\infty}^{\infty} g_n(x)dx$$

where $g_n(x) = \chi_{[-\sqrt{2n}, \sqrt{2n}]}(x)e^x(1 - \frac{x^2}{2n})^n$, $x \in \mathbf{R}$. Now

$$\lim_{t \to \infty} g_n(x) = e^{x - \frac{x^2}{2}} =_{def} h(x)$$

and, as $e^y \geq 1 + y$, $y \in \mathbf{R}$,

$$(1 - \frac{x^2}{2n})^n \leq e^{-\frac{x^2}{2}} \text{ if } -\sqrt{2n} \leq x \leq \sqrt{2n}.$$

Hence,

$$| g_n(x) | \leq h(x),\ \ x \in \mathbf{R},\ \ n \in \mathbf{N}_+$$

where $h \in \mathcal{L}^1(m\ )$ and by using the dominated convergence theorem,

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dt$$

$$= \int_{-\infty}^{\infty} e^{x-\frac{x^2}{2}} dx = e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{2}} dx = e^{\frac{1}{2}} \sqrt{2\pi}.$$

2. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f : X \to \mathbf{R}$ an $(\mathcal{M}$,$\mathcal{R})$-measurable function. Moreover, for each $t > 1$, let

$$a(t) = \sum_{n=-\infty}^{\infty} t^n \mu(t^n \leq | f | < t^{n+1}).$$

Show that

$$\lim_{t\to 1^+} a(t) = \int_X | f | \, d\mu.$$

Solution. Define

$$g_t = \sum_{n=-\infty}^{\infty} t^n \chi_{\{t^n \leq |f| < t^{n+1}\}} \text{ if } t > 1$$

and note that the Beppo Levi theorem implies that

$$\int_X g_t d\mu = a(t).$$

If $| f(x) | = 0$, then $g_t(x) = 0$. Moreover, if $t^n \leq | f(x) | < t^{n+1}$ for some integer n, then $g_t(x) = t^n$ and$| f(x) | \geq g_t(x)$. Thus

$$| f | \geq g_t$$

and we get

$$\int_X | f | \, d\mu \geq \int_X g_t d\mu = a(t).$$

Next suppose $\mid f(x) \mid > 0$ and choose n such that $t^n \leq \mid f(x) \mid < t^{n+1}$. Then

$$tg_t(x) = \sum_{n=-\infty}^{\infty} t^{n+1} \chi_{\{t^n \leq |f| < t^{n+1}\}}(x) = t^{n+1} > \mid f(x) \mid$$

and, hence,

$$tg_t \geq \mid f \mid .$$

Now, by integration,

$$ta(t) \geq \int_X \mid f \mid d\mu.$$

Thus

$$t^{-1} \int_X \mid f \mid d\mu \leq a(t) \leq \int_X \mid f \mid d\mu$$

and

$$\lim_{t \to 1^+} a(t) = \int_X \mid f \mid d\mu.$$

3. Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and $f \in L^1(\mu)$. Define

$$g(t) = \int_X \mid f(x) - t \mid d\mu(x), \ t \in \mathbf{R}.$$

Prove that g is absolutely continuous and

$$g(t) = g(a) + \int_a^t (\mu(f \leq s) - \mu(f \geq s))ds \ \text{ if } a, t \in \mathbf{R}.$$

Solution. Suppose $\varepsilon > 0$ is given and let $]a_k, b_k[$, $k = 1, ..., n$, be disjoint open intervals such that $\Sigma_1^n \mid b_k - a_k \mid < \varepsilon/(1 + \mu(X))$. Then

$$\mid g(a_k) - g(b_k) \mid = \mid \int_X \mid f(x) - a_k \mid - \mid f(x) - b_k \mid d\mu(x) \mid$$

$$\leq \int_X \mid\mid f(x) - a_k \mid - \mid f(x) - b_k \mid\mid d\mu(x)$$

$$\leq \int_X \mid (f(x) - a_k) - (f(x) - b_k) \mid d\mu(x) = \mu(X) \mid b_k - a_k \mid$$

and, consequently,

$$\sum_1^n \mid g(a_k) - g(b_k) \mid \leq \varepsilon.$$

This proves that g is absolutely continuous and therefore g' exists a.e. with respect to Lebesgue measure on $\mathbf{R}$ and

$$g(t) = g(a) + \int_a^t g'(s)ds \text{ for all } t \in \mathbf{R}.$$

Let $A = \{t \in \mathbf{R};\ \mu(f = t) > 0\}$ and note that A is at most denumerable. To compute $g'(s)$ for fixed $s \notin A$, let $(h_n)_0^\infty$ be a sequence of non-zero real numbers which converges to zero. Then

$$\frac{g(s+h_n) - g(s)}{h_n} = \int_X \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} d\mu(x)$$

$$= \int_{\{f \neq s\}} \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} d\mu(x).$$

Here

$$\left| \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} \right| \leq 1$$

and

$$\lim_{n \to \infty} \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} = \begin{cases} 1 \text{ if } s > f(x) \\ -1 \text{ if } s < f(x). \end{cases}$$

Now the dominated convergence theorem gives

$$g'(s) = \int_{\{f \neq s\}} (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu = \int_X (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu$$

$$= \mu(f < s) - \mu(f > s) = \mu(f \leq s) - \mu(f \geq s).$$

In particular,

$$g'(s) = \mu(f \leq s) - \mu(f \geq s)$$

a.e. with respect to Lebesgue measure on $\mathbf{R}$ and since g is absolutely continuous we have

$$g(t) = g(a) + \int_a^t (\mu(f \leq s) - \mu(f \geq s))ds \ \text{ if } a, t \in \mathbf{R}.$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $A_n \in \mathcal{M}$, $n \in \mathbf{N}_+$. Set

$$E = \bigcup_{n \in \mathbf{N}_+} A_n \text{ and } F = \bigcap_{n \in \mathbf{N}_+} A_n.$$

(a) Show that

$$\lim_{n \to \infty} \mu(A_n) = \mu(E)$$

if $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$.

(b) Show that

$$\lim_{n \to \infty} \mu(A_n) = \mu(F)$$

if $\mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$.

5. State and prove the monotone convergence theorem.

Solutions:

INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$,**CTH**$[TMV100]$**)**
January 11, 2010, morning (5 hours), v
No aids.
Examiner: Christer Borell, telephone number 0705292322
Each problem is worth 3 points.

1. Suppose $p \in \mathbf{N}_+$ and define $f_n(x) = n^p x^{p-1}(1-x)^n$, $0 \leq x \leq 1$, for every $n \in \mathbf{N}_+$. Show that

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = (p-1)!.$$

Solution. We have

$$\int_0^1 f_n(x)dx = \left\{ x = \frac{t}{n} \right\} = \int_0^n t^{p-1}(1 - \frac{t}{n})^n dt$$

$$= \int_0^\infty \chi_{[0,n]}(t) t^{p-1} (1 - \frac{t}{n})^n dt.$$

Set $g_n(t) = \chi_{[0,n]}(t) t^{p-1} (1 - \frac{t}{n})^n, t \geq 0$. Then

$$\lim_{t \to \infty} g_n(t) = t^{p-1} e^{-t} =_{def} h(t)$$

and, as $e^y \geq 1 + y$, $y \in \mathbf{R}$, it follows that

$$| g_n(t) | \leq h(t), \; t \geq 0, \; n \in \mathbf{N}_+.$$

Here $h \in \mathcal{L}^1(m \text{ on } [0, \infty[)$, and by using the dominated convergence theorem we have

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = \lim_{n \to \infty} \int_0^\infty g_n(t) dt$$

$$= \int_0^\infty t^{p-1} e^{-t} dt = \Gamma(p) = (p-1)!.$$

2. Let $(X, \mathcal{M}, \mu)$ be a probability space and suppose the sets $A_1, ..., A_n \in \mathcal{M}$ satisfy the inequality $\sum_1^n \mu(A_i) > n - 1$. Show that $\mu(\cap_1^n A_i) > 0$.

Solution. We have

$$\sum_1^n \mu(A_i^c) = \sum_1^n (1 - \mu(A_i)) = n - \sum_1^n \mu(A_i) < n - (n-1) = 1.$$

Hence

$$\mu(\bigcup_1^n A_i^c) \leq \sum_1^n \mu(A_i^c) < 1$$

and

$$\mu(\bigcap_1^n A_i) = \mu((\bigcup_1^n A_i^c)^c) = 1 - \mu(\bigcup_1^n A_i^c) > 0.$$

3. Let μ and ν be probability measures on $(X, \mathcal{M})$ such that $| \mu - \nu | (X) = 2$. Show that $\mu \perp \nu$.

Solution. Set $\sigma = (\mu + \nu)/2$ and note that μ and ν are absolutely continuous with respect to the probability mesure σ. By applying the Radon-Nykodym theorem we get non-negative measurable functions f and g such that $d\mu = fd\sigma$ and $d\nu = gd\sigma$. Here

$$\int_X fd\sigma = \int_X gd\sigma = 1,$$

$$d(\mu - \nu) = (f-g)d\sigma$$

and

$$d \mid \mu - \nu \mid = \mid f - g \mid d\sigma.$$

Now, since $\mid f - g \mid \leq f + g$,

$$2 = \int_X \mid f - g \mid d\sigma \leq \int_X (f+g)d\sigma = 2$$

and we conclude there exists a set $A \in \mathcal{M}$ with $\sigma(A) = 1$ such $f + g = \mid f - g \mid$ on A or, stated otherwise, $(f+g)^2 = \mid f - g \mid^2$ on A. Thus $fg = 0$ on A. Now set $P = \{x \in A;\ f(x) > 0\}$ and $N = P^c$. Then $\mu(P) = 1$ and $\nu(N) = 1$ as $\mu(A^c) = \nu(A^c) = 0$. This proves that $\mu \perp \nu$.

4. State and prove Fatou's Lemma.

5. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Show that $\nu << \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mid \nu(E) \mid < \varepsilon$ whenever $\mu(E) < \delta$.

Solutions:

INTEGRATION THEORY (7.5 hp)

(GU$[MMA110]$,**CTH**$[TMV100]$**)**

August 16, 2010, morning , V.

No aids.

Examiner: Christer Borell, telephone number 0705292322

Each problem is worth 3 points.

1. Let $(X, \mathcal{M}, \mu)$ be a positive measure space, $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of positive real numbers. Set

$$\nu(A) = \sum_{k=1}^n c_k \mu(A \cap E_k),\ \ A \in \mathcal{M}.$$

Show that ν is absolutely continuous with respect to μ and find its Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution. If $A \in \mathcal{M}$ and $\mu(A) = 0$ we have $\mu(A \cap E_k) = 0$ for $k = 1, ..., n$ and it follows that $\nu(A) = 0$. Hence $\nu << \mu$. Moreover, if $A \in \mathcal{M}$,

$$\nu(A) = \sum_{k=1}^n c_k \int_A \chi_{E_k} d\mu = \int_A \sum_{k=1}^n c_k \chi_{E_k} d\mu$$

and thus

$$\frac{d\nu}{d\mu} = \sum_{k=1}^n c_k \chi_{E_k}.$$

2. Suppose $a > 1$. Show that

$$\int_0^\infty \frac{x^{a-1}}{e^{2x} - 1} dx = 2^{-a} \Gamma(a) \varsigma(a)$$

where

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

and

$$\varsigma(a) = \sum_{n=1}^\infty n^{-a}.$$

Solution. We have

$$I =_{def} \int_0^\infty \frac{x^{a-1}}{e^{2x} - 1} dx = 2^{-a} \int_0^\infty \frac{x^{a-1} e^{-x}}{1 - e^{-x}} dx$$

$$= 2^{-a} \int_0^\infty x^{a-1} e^{-x} \sum_{n=0}^\infty e^{-nx} dx.$$

Thus by monotone convergence

$$I = 2^{-a} \sum_{n=0}^\infty \int_0^\infty x^{a-1} e^{-x} e^{-nx} dx$$

$$= 2^{-a} \sum_{n=0}^\infty \frac{1}{(n+1)^a} \int_0^\infty y^{a-1} e^{-y} dy = 2^{-a} \Gamma(a) \sum_{n=0}^\infty \frac{1}{(n+1)^a}$$

$$= 2^{-a} \Gamma(a) \varsigma(a).$$

3. Suppose

$$\mu(A) = \mu_1(A) = \frac{1}{2} \int_A e^{-|t|} dt, \; A \in \mathcal{B}_{\mathbf{R}},$$

$\mu_2 = \mu \times \mu, ...,$ and $\mu_n = \mu_{n-1} \times \mu$, $n \geq 2$. Moreover, let $\varepsilon > 0$ and define

$$A_n = \left\{ x \in \mathbf{R}^n ; \;\; || \, x \, |^2 - 2n \, | \leq \varepsilon n \right\}$$

where $| \, x \, | = \sqrt{x_1^2 + ... + x_n^2}$ if $x = (x_1, ..., x_n) \in \mathbf{R}^n$. Show that

$$\mu_n(A_n^c) \leq \frac{20}{n\varepsilon^2}$$

and conclude that

$$\lim_{n \to \infty} \mu_n(A_n) = 1.$$

Solution. First note that μ_n is a probability measure and

$$\int_{\mathbf{R}} t^2 d\mu(t) = 2.$$

By the Markov inequality

$$\mu_n(A_n^c) \leq \frac{1}{n^2 \varepsilon^2} \int_{\mathbf{R}^n} (| \, x \, |^2 - 2n)^2 d\mu_n(x)$$

$$= \frac{1}{n^2 \varepsilon^2} \int_{\mathbf{R}^n} (\sum_1^n (x_k^2 - 2))^2 d\mu_n(x)$$

$$= \frac{1}{n^2\varepsilon^2}\sum_1^n \int_{\mathbf{R}^n} (x_k^2-2)^2 d\mu_n(x) + \frac{2}{n^2\varepsilon^2}\sum_{1\le j<k\le n}\int_{\mathbf{R}^n}(x_j^2-2)(x_k^2-2)d\mu_n(x).$$

Here, if $j \neq k$,

$$\int_{\mathbf{R}^n}(x_j^2-2)(x_k^2-2)d\mu_n(x) = \int_{\mathbf{R}}(x_j^2-2)d\mu(x_j)\int_{\mathbf{R}}(x_k^2-2)d\mu(x_k) = 0$$

and we get

$$\mu_n(A_n^c) \le \frac{C}{n\varepsilon^2}$$

where

$$C = \int_{\mathbf{R}}(t^2-2)^2 d\mu(t) = 20.$$

Consequently,

$$\lim_{n\to\infty}\mu_n(A_n^c) = 0$$

and since $1-\mu_n(A_n^c) = \mu_n(A_n) \le 1$, we have

$$\lim_{n\to\infty}\mu_n(A_n) = 1.$$

4. Let $\mathcal{C}$ be a collection of open balls in $\mathbf{R}^n$ and let $V = \cup_{B\in\mathcal{C}}B$. Prove that to each $c < m_n(V)$, there exist pairwise disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\Sigma_{i=1}^k m_n(B_i) > 3^{-n}c.$$

(Here m_n denotes Lebesgue measure on $\mathbf{R}^n$.)

5. State and prove the Hahn decomposition theorem.