SOLUTIONS
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$ **,CTH**$[tmv100]$**)**
January 11, 2012, morning, v.
No aids.
Questions on the exam: Fredrik Lindgren 0703 - 088304
Each problem is worth 3 points.
Notation: Lebesgue measure on $\mathbf{R}^n$ is denoted by m_n.

1. Compute the limit

$$\lim_{n\to\infty} \sqrt{n} \int_{-1}^{1} (1-t^2)^n (1+\sqrt{n} \mid \sin t \mid) dt.$$

Solution. We have

$$\sqrt{n} \int_{-1}^{1} (1-t^2)^n (1+\sqrt{n} \mid \sin t \mid) dt = \int_{-\sqrt{n}}^{\sqrt{n}} (1-\frac{x^2}{n})^n (1+\sqrt{n} \mid \sin \frac{x}{\sqrt{n}} \mid) dt$$

$$= \int_{-\infty}^{\infty} f_n(x) dx$$

where

$$f_n(x) = \chi_{[-\sqrt{n},\sqrt{n}]}(x)(1-\frac{x^2}{n})^n((1+\sqrt{n} \mid \sin \frac{x}{\sqrt{n}} \mid).$$

Since

$$e^t \geq 1+t \text{ and } \mid \sin t \mid \leq \mid t \mid \text{ if } t \in \mathbf{R}$$

we get

$$\mid f_n(x) \mid \leq e^{-x^2}(1+ \mid x \mid) \text{ if } x \in \mathbf{R}.$$

Moreover,

$$\lim_{n\to\infty} f_n(x) = e^{-x^2}(1+ \mid x \mid) \in L^1(m)$$

and by dominated convergence,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} e^{-x^2}(1+ \mid x \mid) dx$$

$$= \sqrt{\pi} + 1.$$

2. Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and f and f_n, $n \in \mathbf{N}_+$, measurable functions. Show that $f_n \to f$ in μ-measure if and only if $\arctan(| f_n - f |) \to 0$ in $L^1(\mu)$.

Solution. $\Rightarrow$) First recall that the function arctan is a strictly increasing and continuous function, which vanishes at the origin and is smaller than $\pi/2$.

If $\varepsilon > 0$,

$$0 \leq \int_X \arctan(| f_n - f |) d\mu \leq \int_{|f_n - f| \leq \varepsilon} \arctan(| f_n - f |) d\mu$$

$$+ \int_{|f_n - f| > \varepsilon} \arctan(| f_n - f |) d\mu$$

$$\leq (\arctan \varepsilon) \mu(X) + \frac{\pi}{2} \mu(| f_n - f | > \varepsilon).$$

Hence

$$0 \leq \limsup_{n \to \infty} \int_X \arctan(| f_n - f |) d\mu \leq (\arctan \varepsilon) \mu(X)$$

and since $\varepsilon > 0$ is arbitrary it follows that $\arctan(| f_n - f |) \to 0$ in $L^1(\mu)$.

$\Leftarrow$) Let $\varepsilon > 0$. We have

$$\mu(| f_n - f | > \varepsilon) = \mu(\arctan(| f_n - f |) > \arctan \varepsilon)$$

and the Markov inequality implies that

$$\mu(| f_n - f | > \varepsilon) \leq \frac{1}{\arctan \varepsilon} \int_X \arctan(| f_n - f |) d\mu.$$

Hence

$$\limsup_{n \to \infty} \mu(| f_n - f | > \varepsilon) = 0$$

for every $\varepsilon > 0$ and we conclude that $f_n \to f$ in μ-measure.

3. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f : X \to \mathbf{R}$ a measurable function. Furthermore, suppose there are strictly positive constants B and C such that

$$\int_X e^{af} d\mu \leq Be^{\frac{a^2 C}{2}} \text{ if } a \in \mathbf{R}.$$

Prove that

$$\mu(\mid f \mid \geq x) \leq 2Be^{-\frac{x^2}{2C}} \text{ if } x > 0.$$

Solution. Let $x > 0$ be fixed and note that

$$\mu(\mid f \mid \geq x) = \mu(f \geq x) + \mu(-f \geq x).$$

If, in addition, $a > 0$,

$$\mu(f \geq x) = \mu(e^{af} \geq e^{ax})$$

$$\leq \frac{1}{e^{ax}} \int_X e^{af} d\mu \leq Be^{\frac{a^2 C}{2} - ax}.$$

Now with $a = \frac{x}{C}$,

$$\mu(f \geq x) \leq Be^{-\frac{x^2}{2C}}.$$

Moreover,

$$\int_X e^{a(-f)} d\mu = \int_X e^{(-a)f} d\mu \leq Be^{\frac{a^2 C}{2}} \text{ if } a \in \mathbf{R}$$

and the above gives

$$\mu(-f \geq x) \leq Be^{-\frac{x^2}{2C}}.$$

Thus

$$\mu(\mid f \mid \geq x) \leq 2Be^{-\frac{x^2}{2C}} \text{ if } x > 0.$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, measurable functions such that

$$\sup_{n \in \mathbf{N}_+} \mid f_n \mid \in \mathcal{L}^1(\mu).$$

Moreover, suppose the limit

$$\lim_{n\to\infty} f_n(x)$$

exists and equals $f(x)$ for every $x \in X$.

Prove that $f \in \mathcal{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X \mid f_n - f \mid d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

5. Let $\mathcal{C}$ be a collection of open balls in $\mathbf{R}^n$ and set $V = \cup_{B\in\mathcal{C}} B$. Prove that to each $c < m_n(V)$ there exist disjoint $B_1, ..., B_k \in \mathcal{C}$ such that

$$\sum_{i=1}^{k} m_n(B_i) > 3^{-n} c.$$