INTEGRATION THEORY (7.5 hp)
(**GU**[MMA110] ,**CTH**[tmv100])
August 22, 2012, morning, v.
No aids.
Questions on the exam: Christer Borell 0705 292322
Each problem is worth 3 points.
Notation: Lebesgue measure on $\mathbf{R}^n$ is denoted by m_n.

1. Suppose k and n are non-negative integers such that $k < n$. Show that

$$\int_0^1 \int_0^1 \frac{x^{k+a} y^{n+a}}{1-xy} dxdy = \frac{1}{n-k} \sum_{i=k+1}^{n} \frac{1}{i+a} \text{ if } a > -1.$$

Solution. We have

$$\int_0^1 \int_0^1 \frac{x^{k+a} y^{n+a}}{1-xy} dxdy = \int_0^1 \int_0^1 \sum_{i=0}^{\infty} x^{i+k+a} y^{i+n+a} dxdy$$

and by using the Beppo Levi theorem

$$\int_0^1 \int_0^1 \frac{x^{k+a} y^{n+a}}{1-xy} dxdy = \sum_{i=0}^{\infty} \int_0^1 \int_0^1 x^{i+k+a} y^{i+n+a} dxdy$$

$$= \sum_{i=0}^{\infty} \frac{1}{(i+k+a+1)(i+n+a+1)}.$$

Now, if $L \in \mathbf{N}_+$,

$$\int_0^1 \int_0^1 \frac{x^{k+a} y^{n+a}}{1-xy} dxdy =$$

$$= \frac{1}{n-k} \sum_{i=0}^{L} \left(\frac{1}{i+k+a+1} - \frac{1}{i+n+a+1}\right) + \sum_{i=L+1}^{\infty} \frac{1}{(i+k+a+1)(i+n+a+1)}$$

$$= \frac{1}{n-k} \left(\sum_{i=k}^{n-1} \frac{1}{i+a+1} - \sum_{i=L+k+1}^{L+n} \frac{1}{i+a+1} \right) + \sum_{i=L+1}^{\infty} \frac{1}{(i+k+a+1)(i+n+a+1)},$$

where

$$0 \le \sum_{i=L+k+1}^{L+n} \frac{1}{i+a+1} \le \frac{n-k}{L} \to 0 \text{ as } L \to \infty$$

and

$$0 \le \sum_{i=L+1}^{\infty} \frac{1}{(i+k+a+1)(i+n+a+1)} \le \sum_{i=L+1}^{\infty} \frac{1}{i^2} \to 0 \text{ as } L \to \infty.$$

Hence

$$\int_0^1 \int_0^1 \frac{x^{k+a}y^{n+a}}{1-xy} dxdy = \frac{1}{n-k} \sum_{i=k}^{n-1} \frac{1}{i+a+1} = \frac{1}{n-k} \sum_{i=k+1}^{n} \frac{1}{i+a}.$$

2. Let f:$[0,1] \to]0,\infty[$ be a Borel measurable function. Show that for every $0 < \varepsilon \le 1$ there exists a $\delta > 0$ such that

$$\int_A f(x)dx > \delta$$

for every Lebesgue measurable subset A of $[0,1]$ of Lebesgue measure greater than or equal to ε.

Solution. Suppose $0 < \varepsilon < 1$ and $\mu = m_1$. Since

$$1 = \mu(\cup_{n=1}^{\infty} \left\{ f \ge \frac{1}{n} \right\} = \lim_{n\to\infty} \mu(f \ge \frac{1}{n})$$

there is a positive number c such that

$$\mu(f \ge c) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}\int_A f(x)dx &\ge \int_{A\cap\{f\ge c\}} f(x)dx \ge c\mu(A \cap \{f \ge c\}) \\ &= c(\mu(A) + \mu(f \ge c) - \mu(A \cup \{f \ge c\})) \\ &> c(\mu(A) + 1 - \frac{\varepsilon}{2} - 1) = c(\mu(A) - \frac{\varepsilon}{2})\end{aligned}$$

and

$$\int_A f(x)dx > c\frac{\varepsilon}{2}$$

for every Lebesgue measurable subset A of $[0, 1]$ of μ-measure greater than or equal to ε.

3. Let $A \subseteq \mathbf{R}^n$ be a bounded Lebesgue measurable set of strictly positive Lebesgue measure and denote by k the smallest integer greater than or equal to $m_n(A)$. Moreover, suppose $S = \{(x_1, ..., x_n);\ 0 \leq x_i < 1,\ i = 1, ..., n\}$.

Show that there are k distinct points $j_1, ..., j_k$ of $\mathbf{Z}^n$ and a point $x \in S$ such that

$$x \in \bigcap_{r=1}^{k} (j_r + A).$$

Solution. Since $\mathbf{R}^n$ is a disjoint countable union of the sets

$$j + S,\ j \in \mathbf{Z}^n,$$

we have

$$m_n(A) = \int_{\mathbf{R}^n} \chi_A(x)dx = \sum_{j \in \mathbf{Z}^n} \int_{-j+S} \chi_A(x)dx$$

$$= \sum_{j \in \mathbf{Z}^n} \int_S \chi_A(x - j)dx = \int_S \sum_{j \in \mathbf{Z}^n} \chi_A(x - j)dx.$$

Thus there is a point $x \in S$ such that

$$\sum_{j \in \mathbf{Z}^n} \chi_A(x - j) \geq m_n(A)$$

or

$$\sum_{j \in \mathbf{Z}^n} \chi_A(x - j) \geq k.$$

Now we can find k distinct points $j_1, ..., j_k$ of $\mathbf{Z}^n$ such that $\chi_A(x - j_r) = 1$ for $r = 1, ..., k$, which is proves that

$$x \in \bigcap_{r=1}^{k} (j_r + A).$$

4. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f{:}]a,b[\times X \to \mathbf{R}$ a function such that $f(t,\cdot) \in \mathcal{L}^1(\mu)$ for each $t \in \,]a,b[$ and, moreover, assume $\frac{\partial f}{\partial t}$ exists and

$$\mid \frac{\partial f}{\partial t}(t,x) \mid \leq g(x) \text{ for all } (t,x) \in \,]a,b[\, \times X,$$

where $g \in \mathcal{L}^1(\mu)$. Set

$$F(t) = \int_X f(t,x)d\mu(x) \text{ if } t \in \,]a,b[\,.$$

Prove that F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x) \text{ if } t \in \,]a,b[\,.$$

5. (Egoroff's Theorem) Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and let f_n, $n \in \mathbf{N}_+$, and f be real-valued measurable functions on X such that $f_n \to f$ a.e. $[\mu]$ as $n \to \infty$. Show that for every $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.