Riesz products on $\mathbb{T}$.

$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, though sometimes we replace 2π by 1.

Riesz products are measures defined as $\mu_{(a_j)} = \lim^*_{n\to\infty} \Pi_{j=1}^n (1 + a_j \cos(\lambda_j x)$ where (λ_j) is some lacunary[1] sequence. The sequence (a_j) should consist of integers in the interval $[-1, 1]$.

First of all let us observe that $\mu^n_{(a_j)} = \Pi_{j=1}^n (1 + a_j \cos(\lambda_j x)$ are all positive functions with integral 1 (the later will be shown later), so their L^1-norm is bounded. One can use Riesz representation theorem which identify finite regular measures on X with bounded linear functionals on $C_0(X)$ (in our case, $X = \mathbb{T}$, the later is simply continuous functions). Then by another theorem which you'll learn in the Functional Analysis course any bounded has weak*-limit point (and sometimes even several such points), i.e. when we are talking about $\lim^*$ the difficult part is to show its uniqueness. We will, though, take a shortcut.

By presenting cos in exponential form we see that $\Pi_{j=1}^n (1 + a_j \cos(\lambda_j x) = \sum (\Pi_j (a_j/2)^{\epsilon_j}) exp(x2\pi i \sum \epsilon_j \lambda_j)$ where the sum is taken over all n-tuples of (ϵ_j) where $\epsilon_j = 0, 1, -1$. As the sequence (λ_j) is lacunary there is no simplification of this sum of exponential functions. Thus for every n either there is no presentation of n as $\sum \epsilon_j \lambda_j$ and then $\widehat{\mu^m_{(a_j)}}(n) = 0$ for all k, or there is one and then for all sufficiently large k the Fourier coefficient $\widehat{\mu^m_{(a_j)}}(n)$ is fixed.

Thus, for every trigonometric polynomial $p(x)$ the value $< p, \mu^m_{(a_j)} >$ does not depend from m, as soon as m is large enough, i.e. we can define $\lim_{m\to\infty} < p, \mu^m_{(a_j)} >$ for all trigonometric polynomials.

As trigonometric polynomials are dense in $C(\mathbb{T})$, and the defined functional is obviously bounded we can extend the functional from the trigonometric functions to all of $C(\mathbb{T})$ and thus obtain a unique limit of $\mu^m_{(a_j)}$, call it $\mu_{(a_j)}$.

We want now see whether $\mu_{(a_j)}$ is singular.

Theorem 1. *Let $\mu_{(a_j)}$ and $\mu_{(b_j)}$ are two Riesz products based on the same lacunary sequence $\{\lambda_j\}$. Then $\sum |a_j - b_j|^2 = \infty$ implies that the measures are mutually singular.*

Corollary 2. *If $\sum |a_j|^2 = \infty$ then the Riesz product is singular with respect to Lebesgue measure.*

Proof of the Corollary. Take $b_j = 0$ for all j. Them $\mu_{(b_j)}$ is the Lebesgue measure. $\square$

Proof of the Theorem. Consider $f^a_j = e^{i\lambda_j} - a_j/2$. Those functions are continuous, so $f^a_j \in L^2(\mathbb{T}, \mu_{(a_j)})$.

For $n \neq m$ holds $< f^a_m, f^a_n >= \widehat{\mu_{(a_j)}}(\lambda_m - \lambda_n) - a_m/2\widehat{\mu_{a_j}}(-\lambda_n) - a_n/2\widehat{\mu_{(a_j)}}(\lambda_m) + a_n/2a_m/2 = 0$, where the scalar product is taken in $L^2(\mathbb{T}, \mu_{(a_j)})$. At the same time $\|f^a_j\|^2 =< f^a_j, f^a_j >= \widehat{\mu_{(a_j)}}(0) - a_j/2\widehat{\mu_{(a_j)}}(-\lambda_j) - a_j/2\widehat{\mu_{(a_j)}}(\lambda_j) + (a_j/2)^2 = 1 - (a_j/2)^2 > 3/4$. I.e. the collection of the function f^a_j is orthogonal system, and $\sum c_j f^a_j$ converges in $L^2(\mathbb{T}, \mu_{(a_j)})$ iff $\sum |c_j|^2$ is convergent. One can repeat this argument for (b_j).

[1]Lacunary means that every number can be presented in at most one way by a combination of (λ_j) with coefficients $\{0, 1, -1\}$. In practice it is enough to require $\lambda_{j+1}/\lambda_j \geq 3$ and sometimes one use this last condition as a definition of lacunarity.

Let $\sum |a_j - b_j|^2 = \infty$ then there exists (c_j) such that $\sum c_j(a_j - b_j) = \infty$ and $\sum |c_j|^2 < \infty$. (Proof this as an exercise. Hint: consider separately cases when the sequence $(|a_j - b_j|)$ is bounded or not.)

Consider the series $\sum c_n f_n^a$. It is convergent in $L^2(\mathbb{T}, \mu_{(a_j)})$. By the result about connection of different modes of convergency, we can choose a sequence n_k such that the functions $S_k^a = \sum\limits_{j=1}^{n_k} c_n f_n^a$ converges $\mu_{(a_j)}$-a.e.

Also, $\sum c_n f_n^b$ is convergent in $L^2(\mathbb{T}, \mu_{(b_j)})$. In particular, the S_k^b are convergent in $L^2(\mathbb{T}, \mu_{(b_j)})$, and we can again choose a subsequence $S_{k_m}^b$ which converges $\mu_{(b_j)}$-a.e.

At the same time $S_{k_m}^b = \sum\limits_{j=1}^{n_{k_m}} c_j f_j^b = \sum\limits_{j=1}^{n_{k_m}} c_j f_j^a + \sum\limits_{j=1}^{n_{k_m}} c_j(a_j - b_j) = S_{k_m}^a + \sum\limits_{j=1}^{n_{k_m}} c_j(a_j - b_j)$, where the last term diverges in every point (it is constant). I.e. $S_{k_m}^b$ should diverge in the points in which $S_{k_m}^a$ converges. Take as E the set of points where $S_{k_m}^a$ converge. $\mu_{(a_j)}(\mathbb{T} \setminus E) = 0$, at the same time as $\mu_{(b_j)}(E) = 0$. This shows that the measures are mutually singular. $\square$