Exercises in Functional Analysis, Week V

1. Let X be a Banach space of infinite dimension. Prove that X has no countable "basis" if we let this mean that any vector can be written as a finite linear combination of vectors from the "basis". Hint: If there were a countable basis then X could be written as a union of nowhere dense sets.

2. Let $Y = l^1$ and $X = \{x \in l^1 : \sum_n n|x_n| < \infty\}$ equipped with the l^1-norm.

   (a) Show that X is a proper dense subspace of Y.

   (b) Define $T : X \to Y$ by $(Tx)_n = nx_n$, for $x = (x_1, x_2, \ldots) \in X$. Show that T is closed but not bounded.

   (c) Let $S = T^{-1}$. Show that $S : Y \to X$ is bounded and surjective but not open.

   How this result agrees with the Open Mapping Theorem?

3. Let $Y = C([0, 1])$ and $X = C^1([0, 1])$ (the space of continuously differentiable functions on $[0, 1]$ with one-sided derivatives at the endpoints), both equipped with the uniform norm. X is not complete (see exercise 3, week I). Show that the map $d/dx : X \to Y$, $f \mapsto f'$ is closed but not bounded.

4. Consider the Hilbert space l^2 with the inner product denoted by $\langle \cdot, \cdot \rangle$. To each bounded operator $A \in L(l^2)$ we associate an infinite matrix $(a_{i,j})_{i,j \in \mathbb{N}}$, where $a_{i,j} = \langle Ae_j, e_i \rangle$; here $e_n = (0, \ldots, 0, 1, 0, \ldots)$ (1 in the nth position).

   A function $\varphi : \mathbb{N} \times \mathbb{N}$ is called a **Schur multiplier** if $(\varphi(i, j)a_{i,j})_{i,j \in \mathbb{N}}$ is the matrix of a bounded operator whenever so is $(a_{i,j})$.

   Prove that φ is a Schur multiplier if and only if the mapping $S_\varphi : L(l^2) \to L(l^2)$ which sends $A \in L(l^2)$ to $S_\varphi(A)$, $\langle S_\varphi(A)e_j, e_i \rangle = \varphi(i, j)\langle Ae_j, e_i \rangle$, is a bounded operator on $L(l^2)$ equipped with the operator norm. Hint: Use the Closed Graph theorem.

5. Let H be a Hilbert space and let $A : H \to H$ be a linear operator such that there exists a linear operator $B : H \to H$ such that

   $$\langle Ax, y \rangle = \langle x, By \rangle \text{ for all } x, y \in H.$$

   Show that A is bounded. Hint: Show that the graph of A is closed.

6. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Assume $f_i \to f$ in L^p, and $g_i \to g$ weakly in L^q. Show that $f_i g_i \to fg$ weakly in L^1.

7. Let $X = C^1([0, 1])$ with the uniform norm. Define $A_n : X \to C([0, 1])$ by

   $$(A_n x)(t) = n(x(t + \frac{1}{n}) - x(t)), t \in [0, 1], x \in X, n \geq 1,$$

   (if $t > 1$ we let $x(t) := x(1) + x'_-(1)(t - 1)$). Show that

(a) $\{A_n x : n \geq 1\}$ converges for any $x \in X$;

(b) $\{\|A_n\|\}$ is unbounded;

(c) the space $L(X, C([0,1]))$ is not complete with respect to the strong operator convergence.

8. Let $1 \leq p, q \leq \infty$ be conjugate exponents.

   (a) Let $a = (a_1, a_2, \ldots)$ be a sequence such that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for any $x = (x_1, x_2, \ldots)$ in l^p. Prove that $a \in l^q$.

   (b) Let p be a measurable function such that the functional $f(x) = \int_a^b p(t)x(t)dt$ is defined for all $x \in L^p([a,b])$. Show that $L^q([a,b])$.

   Hint: Use the Uniform Boundedness Principle.