

Exercises in Functional Analysis, Week VII

1. Let H be a Hilbert space and $M, N, M \subset N$, be subsets of H . Show that
 - (a) $M^\perp \supset N^\perp$;
 - (b) $(M^\perp)^\perp = \overline{\text{span}\{M\}}$, the closed linear span of M .
2. Find the orthogonal complement in $L^2([0, 1])$ to the following sets:
 - (a) all polynomials;
 - (b) all polynomials p such that $p(0) = 0$.
3. Let H be a Hilbert space, and $G \subset H$ a closed subspace. Prove that any linear bounded functional on G has a unique extension on H that preserves the norm. More precisely:
 - (a) If $f : G \rightarrow \mathbb{C}$ is a linear bounded functional, then $\tilde{f} : H \rightarrow \mathbb{C}$ defined by $\tilde{f}(x) = f(P_G(x)) \forall x \in H$ is the unique extension of f to H such that $\|f\| = \|\tilde{f}\|$, where P_G is the orthogonal projection onto G .
 - (b) If $b \in H$ and $f : G \rightarrow \mathbb{C}$ is defined by $f(x) = \langle x, b \rangle$, $x \in G$, then the unique extension preserving the norm for f is $\tilde{f} : H \rightarrow \mathbb{C}$ defined by $\tilde{f}(x) = \langle x, P_G(b) \rangle$, $x \in H$, and $\|f\| = \|\tilde{f}\| = \|P_G(b)\|$.
4. Let $B = \text{span}\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$. For $x, y \in B$ define

$$\langle x, y \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \overline{y(t)} dt.$$

Show that

- (a) $\langle \cdot, \cdot \rangle$ is an inner product on B ;
 - (b) the system $\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$ is orthonormal. Conclude that the completion of B with respect to the given inner product is a non-separable Hilbert space.
5. Let H be a Hilbert space. Show that every weakly closed set is closed in the norm topology but the converse is not true. Show that if we assume that the set is a closed subspace of H then it is weakly closed.
 6. Let H be a Hilbert space. Show that the norm convergence is the same as weak convergence uniformly on the unit sphere, i.e. $\|f_n - f\| \rightarrow 0$ if and only if $\langle f_n, g \rangle \rightarrow \langle f, g \rangle$ uniformly for $\|g\| = 1$.
 7. Prove that $l \in C([a, b])^*$ and find $\|l\|$ if

(a) $l(x) = \alpha x(a) + \beta x(b)$, $(\alpha, \beta \in \mathbb{R})$;

(b) $l(x) = \int_a^b x(t)p(t)dt$, $p \in C([a, b])$;

(c) $l(x) = \int_a^b x(t)dt - x\left(\frac{a+b}{2}\right)$.

8. Show that $l(g) = \int_0^{1/2} dg(t)$ is a linear bounded functional on $C([0, 1])^*$. Conclude that $C([0, 1])$ is not reflexive.