Exercises in Functional Analysis, Week I

1. Let $X \neq \{0\}$ be a real or complex linear space. Prove that there is at least one norm on X.

2. (a) Prove that the formula
$$\langle A, B\rangle = \mathrm{trace}(B^* A)$$
defines an inner product on the space $M_{m\times n}(\mathbb{C})$ of $m \times n$-matrices. By the trace of a $n \times n$-matrix A we mean the sum of its diagonal elements.

   (b) What is $||A||_2$ for $A \in M_{n\times n}$? Let $A = A^*$. Show that $\|A\|_2 = \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}$ where λ_i are the A:s eigenvalues counted with their multiplicities. Hint: Show that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ for any pair of $n \times n$ matrices A and B.

3. Consider the Banach space l^1. Show that the parallelogram law does not hold in l^1. Conclude that $\|\cdot\|_1$ is not induced by some inner product.

4. Show that equivalent norms define the same open sets and Cauchy sequences.

5. Consider the space $X = C^1([0,1])$ of continuously differentiable functions on $[0,1]$ with one-sided derivatives at the endpoints and equip X with the norm
$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}.$$
Show that X is not a Banach space (see exercise 5.1.9 in Folland).

6. Take the underlying vector space $X = C([0,1]$ with two different norms
$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$$
and
$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}.$$
Consider the sequence
$$f_n(t) = \begin{cases} -nt + 1, & t \in [0, \frac{1}{n}] \\ 0, & t \in [\frac{1}{n}, 1] \end{cases}$$
Show that the sequence converges to $g \equiv 0$ in $(C([0,1]), \|\cdot\|_2)$ but not in $(C([0,1]), \|\cdot\|_\infty)$. Conclude that the given norms are inequivalent on $C([0,1])$ (See Example 6.7 in Lecture Notes). Is the sequence convergent in $(C([0,1]), \|\cdot\|_\infty)$.

7. Which of the following sequences are convergent in the respective normed spaces? For convergent sequences find the limit.

- $(C([0,1]), \|\cdot\|_\infty)$, $f_n(t) = \sin^n \pi t$, $t \in [0,1]$;
- $(C([0,1]), \|\cdot\|_\infty)$, $f_n(t) = \frac{nt}{1+n^\alpha t^2}$, where $\alpha \geq 1$ is fixed;
- $(l_p, ||\cdot||_p)$, $1 \leq p \leq \infty$, $x^{(n)} = (\underbrace{0,\ldots,0}_{n-1}, 1, 0, \ldots)$.

8. Show that the vector space $C([0,1])$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$ is an inner-product space which is incomplete with respect to the norm $\|f\|_2 = \langle f, f\rangle^{1/2}$ and so is not a Hilbert space.

9. Show that if $\mathcal{X}$ is a real Banach space then $\mathcal{X} \times \mathcal{X}$ becomes a complex Banach space $\mathcal{X}_{\mathbb{C}}$ when its linear structure and norm are defined by

$$\begin{aligned}(x,y) + (u,v) &= (x+u, y+v)\\ (a+ib)(x,y) &= (ax - by, bx + ay)\\ \|(x,y)\| &= \sup\{\|(\cos\theta)x + (\sin\theta)y : 0 \leq \theta \leq 2\pi\|\}\end{aligned}$$

for all x, y, u, $v \in \mathcal{X}$ and a, $b \in \mathbb{R}$.

10. (a) Prove that any ball B (open or closed) in a normed vector space is convex, i.e. that $\lambda x + (1-\lambda)y \in B$ whenever x, $y \in B$, $0 \leq \lambda \leq 1$.

(b) Consider the real two-dimensional l_2^p, i.e. $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}$. Sketch the unit ball for several values of p, including $p = 1, 3/2, 2, 3, \infty$. For which p:s is the closed unit ball strictly convex?(i.e. for which p:s does $\|x\|$, $\|y\| \leq 1$, $x \neq y$ and $\lambda \in (0,1)$ imply that $\|\lambda x + (1-\lambda)y\| < 1$?)

(c) Prove that the triangle axiom from the definition of a norm is equivalent with the convexity of the closed unit ball. More precisely, if X is a linear space on which is given a function $p : X \to [0, \infty)$ with the properties: (1) $p(x) = 0 \Leftrightarrow x = 0$; (2) $p(\lambda x) = |\lambda| p(x)$ $\forall x \in X$, $\forall \lambda \in \mathbb{C}$, then p is a norm if and only if the closed unit ball is convex.

11. Consider the spaces l^1, l^2 and l^∞. For which of these normed vector spaces is it true that $\|x+y\| = \|x\| + \|y\|$ implies that x and y are parallell (i.e.one is a scalar multiple of the other).

12. Show that all norms $\|\cdot\|$ on a finite-dimensional vector space X are equivalent, using the following arguments: let $e_1, \ldots, e_n$ be a basis and define the norm $\|\sum_{i=1}^n a_i e_i\|_1 = \sum_{i=1}^n |a_i|$. First show that there is a C such that $\|x\| \leq C\|x\|_1$ for all $x \in X$. From this conclude that $\|\cdot\|$ is bounded away from 0 on the compact set $\{x : \|x\|_1 = 1\}$ of X with the topology defined by $\|\cdot\|_1$. Finally conclude from this that there exists C' such that $\|x\|_1 \leq C'\|x\|$ for all $x \in X$. (see exercise 5.6 in Folland).

13. Show that every finite-dimensional subspace, V, of a normed vector space, X, is closed.