Exercises in Functional Analysis, Week II

1. Let $p \in (0,1)$. Show, using characteristic functions of disjoint sets that $\| \cdot \|_p$ is not a norm on $L^p([0,1])$. Show also that the "unit ball" $\{f : \|f\|_p \leq 1\}$ is not convex.

2. Let $1 \leq p \leq \infty$. Prove that the normed spaces $L^p(\mathbb{R})$, $L^p(0,1)$ are isometrically isomorphic, i.e. there exists a bijective linear map $I : L^p(\mathbb{R}) \to L^p(0,1)$ such that $\|I(f)\|_p = \|f\|_p$, $f \in L^p(\mathbb{R})$.

3. Show that for $1 \leq p < q < \infty$ neither $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$ nor $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$.

4. (Packing of balls) Let $1 \leq p \leq \infty$. Find, inside the unit ball of l^p, infinitely many pairwise disjoint balls of the same positive radius. Conclude from this that the closed unit ball in l^p cannot be compact.

5. Let M be a complete metric space and K a subset of M. Show that the following are equivalent

   (a) K is compact, i.e. every open cover of K can be reduced to a finite subcover.

   (b) K is sequentially compact, i.e. every sequence in K has a convergent subsequence.

6. (a) Let X be an infinite-dimensional normed vector space. Show that for sufficiently small $r > 0$ one can find infinitely many pairwise disjoint balls of radius r inside the unit ball.

   (b) Make the conclusion that a ball in X is compact if and only if X is finite dimensional.

7. Prove that l^∞ and $L^\infty([0,1])$ are not separable, i.e. they does not contain a countable dense subset. Hint: Find uncountable set of points, all at mutual distance at least r for some $r > 0$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that μ is separable if there exists a countable family $\mathfrak{A} = \{A_1, A_2, \ldots\}$ of measurable sets such that

$$(\forall \varepsilon > 0)(\forall B \in \mathcal{M})(\exists A_k \in \mathfrak{A}) : \mu(A_k \triangle B) < \varepsilon,$$

   here $\triangle$ denotes the symmetric difference: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

   (a) Show that $L^p(X, \mu)$, $1 \leq p < \infty$, is a separable Banach space if μ is a separable measure.

   (b) Show that $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is separable (the measure is the Lebesgue measure. Observe that the Lebesque measure on $\mathbb{R}^n$ is not separable!).

9. Define a mapping $T : C([0,1] \to \mathbb{C}$ by the formula

$$T(f) = \int_0^1 f(t)t^2 dt.$$

Prove that T is a bounded linear functional with respect to the norm $\|f\|_p$, $1 < p < \infty$, where $\|f\|_p = \left(\int_0^1 |f(t)|^p\right)^{1/p}$. Find the norm of this functional.

10. Let $X = l_2$, $f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k}(x_k + x_{k+1})$, $x \in l^2$. Show that f is a linear bounded functional on X and find its norm.

11. Let $F : C^1([0,1]) \to \mathbb{C}$ be defined by $F(f) = f'(1)$, $f \in C^1([0,1])$. Show that F is a linear functional. Prove that F is discontinuous with respect to $\|f\|_2$.

12. Set $(Ax)(t) = x'(t)$ and $(Bx)(t) = tx(t)$, $0 < t < 1$, for $x \in C^\infty(0,1)$. Prove that $AB - BA = I$. Prove that it is impossible to find a norm on $C^\infty(0,1)$ such that A and B are bounded operators with respect to this norm by proving the following general result: If E be a non-trivial normed space then there are no bounded linear operators A and B on E such that $AB - BA = I$. Hint: Show that $A^n B - BA^n = nA^{n-1}$.

13. (a) Let X be a finite-dimensional space. Find a general form of a linear functional on X. Show that any linear functional on this space is continuous.

    (b) Show that any linear functional on $M_n(\mathbb{C})$ is given by $f(A) = \text{trace}(AB)$ for all $A \in M_n(\mathbb{C})$ and some $B \in M_n(\mathbb{C})$. What is the norm of this linear functional if $M_n(\mathbb{C})$ is equipped with the Hilbert-Schmidt norm $\|A\|_2 = \text{trace}(A^*A)^{1/2}$?