Exercises in Functional Analysis, Week III

1. Suppose that X is a normed space and f is a linear functional on X. Let $\ker f = \{x : f(x) = 0\}$. Prove that

   (a) $\ker f$ is a subspace of X;

   (b) $\ker f$ is a hyperplane if f is not identically zero (a linear subspace M of a vector space is called a hyperplane if for some (and hence all) $x \in X \setminus M$ the subspace M and the vector x together span X. In this case one says that M has codimension 1);

   (c) any subspace of codimension 1 is the kernel of a linear functional;

   (d) $\ker f$ is closed if and only if f is bounded (Hint: for "only if" use the result in (b));

   (e) if two linear functionals have the same kernel then they are proportional, i.e. $f = \alpha g$ for some constant α.

2. For which $\alpha \in \mathbb{R}$ the following linear functionals are bounded? For bounded functionals find the norms.

   (a) $X = l^p$, $1 \le p \le \infty$, $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n^\alpha}$;

   (b) $X = l^p$, $1 \le p \le \infty$, $f(x) = \displaystyle\sum_{n=1}^{\infty} x_n((n+1)^\alpha - n^\alpha)$;

3. Consider the normed space $(c_0, \|\cdot\|_\infty)$, where

$$c_0 = \{\underline{x} = (x_1, \dots, x_i, \dots) : x_i \in \mathbb{C},\ x_i \to 0 \text{ as } i \to \infty\}$$

   and $\|\underline{x}\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$. Prove that c_0^* is isometrically isomorphic to l^1 by proving that any continuous linear functional f on c_0 is given by

$$f(\underline{x}) = \sum_{i=1}^{\infty} x_i a_i, \underline{x} \in c_0$$

   for some $\underline{a} = (a_1, \dots, a_i, \dots)$ in l^1 and that $\|f\| = \|\underline{a}\|_1$.

4. Consider the normed space $(c, \|\cdot\|_\infty)$, where

$$c = \{\underline{x} = (x_1, \dots, x_i, \dots) : x_i \in \mathbb{C},\ \exists \lim_{i \to \infty} x_i\}$$

and $\|\underline{x}\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$. Prove that c^* is isometrically isomorphic to l^1 by proving that any continuous linear functional f on c is given by

$$f(\underline{x}) = a_0 \lim_{i\to\infty} x_i + \sum_{i=1}^{\infty} x_i a_i, \underline{x} \in c$$

for some $\underline{a} = (a_0, a_1, \ldots, a_i, \ldots)$ in l^1 and that $\|f\| = \|\underline{a}\|_1$. Deduce that neither of Banach spaces c_0 and c is isomorphic with its double dual.

5. A linear functional $f : C([a,b]) \to \mathbb{R}$ is called positive if $f(x) \geq 0$ for any $x(t) \in C([a,b])$ such that $x(t) \geq 0$, $t \in [a,b]$. Show that

   (a) any positive linear functional is continuous and $\|f\| = f(1)$ where 1 denotes the constant function $x(t) = 1$;

   (b) if $f \in (C([a,b])^*$ and $\|f\| = f(1)$ then f is a positive linear functional (Hint: write any $x \in C([a,b])$, $0 \leq x \leq 1$ as $x = 1 - (1-x)$ and observe that $1 - x$ is in the unit ball of $C([a,b])$).

6. (Extension by continuity) Let M be a dense subspace of the normed space X. Then clearly the restriction to M of the norm of X makes M into a normed space. Let Y be a Banach space. Show that any bounded linear operator $T : M \to Y$ has a unique extension to a bounded linear operator $X \to Y$.