Exercises in Functional Analysis, Week IV

1. The map $(s, 0) \mapsto s$ is a linear functional on the subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2$. Determine all its linear extensions to $\mathbb{R}^2$ with the same norm if $\mathbb{R}^2$ is given
   (a) the l^1 norm;
   (b) the l^p norm, $1 < p \leq \infty$.

2. Show that there exists a non-zero linear functional $F \in (L^\infty([a, b]))^*$ such that for any $f \in C([a, b])$, $F(f) = f((a + b)/2)$. Prove that the inclusion map $T : L^1([a, b]) \to (L^\infty([a, b]))^*$, $f \mapsto \phi_f$, where $\phi_f(g) = \int_a^b f(t)g(t)dt$, is not surjective.

3. Consider l^2_F, the linear subspace of l^2 consisting of those sequences having only finitely many terms different from zero. Describe the space $(l^2_F)^*$.

4. Prove that there exists a linear functional $F \in (l^\infty)^*$ such that $F(x) = \lim x_n$ for any $x \in c$, where c is the linear subspace of l^∞ consisting of convergent sequences. Show that the inclusion $T : l^1 \to (l^\infty)^*$, $a \mapsto \phi_a$, $\phi_a(x) = \sum_{i=1}^\infty x_i a_i$, is proper, i.e. $T(l^1)$ is a proper subspace of $(l^\infty)^*$.

5. Let $\{x_1, x_2, \ldots, x_n\}$ be a system of linearly independent elements of a normed space X. Let $c_1, \ldots, c_n \in \mathbb{C}$. Show that there exists $f \in X^*$ such that $f(x_i) = c_i$.

6. Let $\{x_n\}$ be a sequence of elements in a Banach space X. Let L be a subspace of all its finite linear combinations and $\overline{L}$ is the closure of L. Show that $x \in \overline{L}$ if and only if for any $f \in X^*$ such that $f(x_n) = 0$ one has $f(x) = 0$.

7. Is it true that two elements x and y of a normed linear space X are equal iff $f(x) = f(y)$
   (a) for all $f \in X^*$;
   (b) for all f in a dense subset of X^*.

8. Let l^∞ be a real Banach space. Let $G = \{(x_1, x_2 - x_1, x_3 - x_2, \ldots) : x = (x_1, x_2, \ldots) \in l^\infty\}$ and $e = (1, 1, \ldots)$. Show that there exists $f \in (l^\infty)^*$ such that $f(x) = 0$, $x \in \bar{G}$, $f(e) = 1$, $\|f\| = 1$ (Hint: Show that the distance from e to the closure of G equals 1). The functional is denoted by $LIMx_n := f(x)$, $x \in l^\infty$. Show that
   (a) $LIMx_n = LIMx_{n+1}$, i.e. $f(x) = f(S(x))$, where S is the shift operator: $S((x_1, x_2, \ldots)) = (x_2, x_3, \ldots)$.
   (b) $LIMx_n = a$ if $x_n \to a$;
   (c) there exists $x \in l^\infty$ such that for any $N \geq 1$ there exists $n \geq N$ such that $x_n = 1$ and $LIMx_n = 0$.

9. Let X be a normed space, $G \subseteq X$ is a linear space, and $f : G \to \mathbb{C}$ is a bounded linear functional for which there exist $g, h : X \to \mathbb{C}$, two distinct linear bounded extensions such that $\|g\| = \|h\| = \|f\|$. Prove that f has infinity of linear bounded extensions of norm $\|f\|$. More precisely, one can prove that the set of all such extensions is convex in X^*.

10. (a) Let $1 < p < \infty$ and $G = \{(x_n)_{n\in\mathbb{N}} \in l^p : \sum_{n=1}^{\infty} x_n = 0\}$. Use a consequence from the Hahn-Banach theorem to prove that G is a dense linear subspace of l^p.

    (b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of scalars such that $\sum_{n=1}^{\infty} |a_n| \neq 0$ and let $1 < p < \infty$. Find a necessary and sufficient condition on the sequence $(a_n)_{n\in\mathbb{N}}$ for the linear subspace $G = \{(x_n)_{n\in\mathbb{N}} \in l^p : \sum_{n=1}^{\infty} a_n x_n = 0\}$ be dense in l^p.

11. (Folland, exercise 25, chapter 5.2) Let X be a real vector space and let P be a subset of X such that (i) if x, $y \in P$, then $x + y \in P$, (ii) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$, (iii) if $x \in P$ and $-x \in P$, then $x = 0$. (Example: if X is a space of real valued functions, P can be the set of nonnegative functions in X). Show

    - The relation $\leq$ defined by $x \leq y$ iff $y - x \in P$ is a partial ordering on X.
    - (Krein's extension theorem) Suppose that M is a subspace of X such that for each $x \in X$ there exists $y \in M$ with $x \leq y$. If f is a linear functional on M such that $f(x) \geq 0$ for $x \in P \cap M$, there is a linear functional on M such that $F(x) \geq 0$ for $x \in P$ and $F|_M = f$. (Consider $p(x) = \inf\{f(y) : y \in M, x \leq y\}$).