Exercises in Functional Analysis, Week VI

1. Let $\{x_n\}$ be a sequence in $L^p([a,b])$, $x \in L^p([a,b])$, $1 < p < \infty$. Show that $x_n \to x$ weakly iff $\sup_n \|x_n\| < \infty$ and $\int_a^\tau x_n(t)dt \to \int_a^\tau x(t)dt$ for all $\tau \in [a,b]$.

   Consider the function sequence $\{e^{inx}\}_{n\in\mathbb{N}}$ in $L^p([-\pi,\pi])$. For which p, $1 \le p \le \infty$, is it weakly or weakly* convergent? (For L^1 you consider only weak convergence; use $L^p \simeq (L^q)^*$, for $p > 1$, where q, as usually, the conjugate exponent to p).

2. Let (e_n) denote the "standard basis" in ℓ^p. For which p with $1 \le p \le \infty$ does e_n converge weakly or weakly* as $n \to \infty$, and what is the limit? (Be extra careful with the cases $p = 1$, $p = \infty$, consider the duality $c_0^* \simeq \ell^1$ and $c^* \simeq \ell^1$ for $p = 1$, and $\ell^p \simeq (\ell^q)^*$ for $p > 1$).

3. We know that $\ell^1 \simeq c_0^*$ and $(\ell^1)^* \simeq \ell^\infty$. Give an example of a sequence of vectors in ℓ^1 that converges weakly* but not weakly, which means that $f(x_n) \to f(x)$ for all $f \in c_0$ but not for all $f \in \ell^\infty$.

4. (a) Prove that translation is continuous in $L^p(\mathbb{R})$ for $1 \le p < \infty$: If $f_t(x) = f(x-t)$ with $f \in L^p$ then $f_t \to f_{t_0}$ in L^p as $t \to t_0$. (Hint: use the fact that the space of continuous functions with compact support, i.e. continuous functions which are zero outside of a compact set in $\mathbb{R}$, are dense in L^p). Prove also that this is false for $p = \infty$.

   (b) Show that in L^∞ the translation is weakly* continuous, i.e. show that $f_t \to f_{t_0}$ weakly* in $L^\infty = (L^1)^*$.

5. Let X be a Banach space and X^* its dual. Assume that $\{x_j\}$ is a sequence in X that converges to $x \in X$, either in norm or weakly. Similarly , $\{\varphi_j\}$ is a sequence in X^* with a limit φ, and the convergence is assumed to be either in norm or weak*. This gives four different cases of convergence assumption. The question is: does it follow that $\varphi_j(x_j) \to \varphi(x)$? Answer this question in each of the four cases, by means of a proof or a counterexample. Hint: Use that a sequence that converges weakly or weakly* is bounded in norm.

6. Let H be a Hilbert space, x, $x_n \in H$, $n \ge 1$. Show that $x_n \to x$ in H if $x_n \to x$ weakly and either (a) $\|x_n\| \to \|x\|$ or (b) $\limsup \|x_n\| \le \|x\|$.

7. Let X be a reflexive Banach space and $\{x_n\} \subset X$. Prove that if $\{f(x_n)\}$ is Cauchy for all $f \in X^*$ then there exists $x \in X$ such that $x_n \to x$ weakly.

8. (An application of Banach-Alaoglu theorem): Any normed space X can be isometrically embedded into $C(\Omega)$, the Banach space of continuous functions on a compact Ω. Hint: Take $\Omega = \{f \in X^* : \|f\| \le 1\}$ and consider the mapping $X \ni x \mapsto \hat{x}|_\Omega \in C(\Omega)$.

9. Let X be a reflexive Banach space. Show that any bounded sequence of elements in X has a weakly convergent subsequence. Hint: Use the following facts: (1) If X^* is separable then so is X (in their respective norm topologies); (2) Closed subspaces of reflexive spaces are reflexive.