Exercises in Functional Analysis, Week VII

1. Let H be a Hilbert space and M, N, $M \subset N$, be subsets of H. Show that
   (a) $M^{\perp} \supset N^{\perp}$;
   (b) $(M^{\perp})^{\perp} = \overline{\text{span}\{M\}}$, the closed linear span of M.

2. Find the orthogonal complement in $L^2([0,1])$ to the following sets:
   (a) all polynomials;
   (b) all polynomials p such that $p(0) = 0$.

3. Let H be a Hilbert space, and $G \subset H$ a closed subspace. Prove that any linear bounded functional on G has a unique extension on H that preserves the norm. More precisely:
   (a) If $f : G \to \mathbb{C}$ is a linear bounded functional, then $\tilde{f} : H \to \mathbb{C}$ defined by $\tilde{f}(x) = f(P_G(x))$ $\forall x \in H$ is the unique extension of f to H such that $\|f\| = \|\tilde{f}\|$, where P_G is the orthogonal projection onto G.
   (b) If $b \in H$ and $f : G \to \mathbb{C}$ is defined by $f(x) = \langle x, b\rangle$, $x \in G$, then the unique extension preserving the norm for f is $\tilde{f} : H \to \mathbb{C}$ defined by $\tilde{f}(x) = \langle x, P_G(b)\rangle$, $x \in H$, and $\|f\| = \|\tilde{f}\| = \|P_G(b)\|$.

4. Let $B = \text{span}\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$. For x, $y \in B$ define
$$\langle x, y\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\overline{y(t)}dt.$$
   Show that
   (a) $\langle \cdot, \cdot \rangle$ is an inner product on B;
   (b) the system $\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$ is orthonormal. Conclude that the completion of B with respect to the given inner product is a non-separable Hilbert space.

5. Let P and Q denote orthogonal projections onto two subspaces in a Hilbert space. Prove that $\|P - Q\| \leq 1$.

6. Let H be a Hilbert space. Show that every weakly closed set is closed in the norm topology but the converse is not true. Show that if we assume that the set is a closed subspace of H then it is weakly closed.

7. Let H be a Hilbert space. Show that the norm convergence is the same as weak convergence uniformly on the unit sphere, i.e. $\|f_n - f\| \to 0$ if and only if $\langle f_n, g\rangle \to \langle f, g\rangle$ uniformly for $\|g\| = 1$.

8. Prove that $l \in C([a,b])^*$ and find $\|l\|$ if

   (a) $l(x) = \alpha x(a) + \beta x(b)$, $(\alpha,\ \beta \in \mathbb{R})$;

   (b) $l(x) = \int_a^b x(t)p(t)dt$, $p \in C([a,b])$;

   (c) $l(x) = \int_a^b x(t)dt - x\left(\frac{a+b}{2}\right)$.

9. Show that $l(g) = \int_0^{1/2} dg(t)$ is a linear bounded functional on $C([0,1])^*$. Conclude that $C([0,1])$ is not reflexive.