Questions to the oral exam in Functional Analysis, 2017

1. Normed spaces, Banach spaces: definitions, examples, importance. Prove that a complete linear subspace of a Banach space is closed. Prove that a closed subspace in a Banach space is complete.
2. Hölder's inequality. Its role. Minkowski's inequality. Its role. L^p-spaces, relationships between different L^p-spaces.
3. Pre-Hilbert spaces. Definition, examples, main properties, parallellogram identity.
4. Separable spaces. Definition, examples. Prove that ℓ^∞ is not separable, but ℓ^p, $1 \le p < \infty$, is separable. What about L^p spaces?
5. Show that the space of bounded mappings $B(X, Y)$, where X is a normed space and Y is Banach, is a Banach space. Argue why dual spaces are complete.
6. Linear functionals, norms of linear functionals. Equivalent conditions of boundedness of linear functionals.
7. Dual of L^p-spaces.
8. The Hahn-Banach theorem for real and complex linear functionals. The structure of the proofs.
9. Corollaries from the Hahn-Banach theorem. The second dual. Reflexivity of a Banach space. Examples.
10. Dual of ℓ^∞.
11. The Baire category theorem.
12. The open mapping theorem. Banach Theorem about bounded inverse. Closed Graph Theorem.
13. The uniform boundedness principle (Banach-Steinhaus). Importance and applications.
14. Weak convergence. Definition, boundedness, other properties. Examples in ℓ_p, L^p and $C([0, 1])$.
15. Weak*-convergence. Examples. Examples that show the difference. Banach-Alaoglu theorem. Alaouglu theorem.
16. Hilbert spaces, definition, main properties. Theorem on the closest element.
17. Hilbert spaces: theorem on decomposition of a Hilbert space.

18. Riesz-Frechet theorem.
19. Orthogonal systems in Hilbert spaces, Bessel inequality. Definition of orthonormal basis, equivalent conditions. Theorem on existence of orthonormal basis in each Hilbert space.
20. The Riesz Representation Theorem on bounded linear functionals on $C([a, b])$.
21. The space of linear bounded operators: adjoint operators; spectrum (if we have time for this!)