# Exercises in Spectral Theory and Operator Algebras

1. Consider the Banach space $C[0,1]$ of continuous complex-valued functions on the unit interval with sup norm.

   (a) Let $k(x,y)$ be a continuous function defined on the triangle $0 \leq y \leq x \leq 1$. Consider an integral map $K$ defined on $C[0,1]$ as follows
   $$Kf(x) = \int_0^x k(x,y)f(y)dy, f \in C[0,1].$$
   Show that $K$ defines a bounded operator on $C[0,1]$.

   (b) For the kernel $k(x,y) = 1$, $0 \leq y \leq x \leq 1$ consider the corresponding Volterra operator $V : C[0,1] \to C[0,1]$,
   $$Kf(x) = \int_0^x f(y)dy, f \in C[0,1].$$
   Given a function $g \in C[0,1]$, show that the equation $Vf = g$ has a solution $f \in C[0,1]$ iff $g$ is continuously differentiable and $g(0) = 0$.

   (c) Let $k(x,y)$ be a continuous function defined on the unit square $0 \leq x, y \leq 1$. Consider an integral map $K$ defined on $C[0,1]$ by
   $$Kf(x) = \int_0^1 k(x,y)f(y)dy, f \in C[0,1].$$
   Let $B_1$ be the closed unit ball in $C[0,1]$. Show that $K$ is a compact operator in the sense that the norm closure of the image $KB_1$ is a compact subset of $C[0,1]$.

2. Let $A$, $B$ be shift operators on $\ell^2$ defined by
   $$A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$
   $$B(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$
   for $x = (x_1, x_2, \ldots) \in \ell^2$. Show that $\|A\| = \|B\| = 1$, and compute both $BA$ and $AB$. Deduce that $A$ is injective but not surjective, $B$ is surjective but not injective, and that $\sigma(AB) \neq \sigma(BA)$.

3. Let $E$ be a Banach space and let $A$ and $B$ be bounded operators on $E$. Show that $1 - AB$ is invertible if and only if $1 - BA$ is invertible. Hint: Think about how to relate the formal Neumann series for $(1 - AB)^{-1}$,
   $$(1 - AB)^{-1} = 1 + AB + (AB)^2 + \ldots$$
   to that for $(1 - BA)^{-1}$ and turn your idea into a rigorous proof.

4. Use the result of the preceding exercise to show that for any two bounded operators $A$, $B$ acting on a Banach space, $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$.

5. Let $T$ be a bounded operator on a Banach space $E$.

   - Let $\lambda \in \sigma(T)$. Assume that there exists $C > 0$ such that $\|Tx - \lambda x\| \geq C\|x\|$ for all $x \in E$. Show that $\lambda \in \sigma_r(T)$, the residual spectrum of $T$.
   - Let $\lambda \in \sigma_c(T)$, the compression spectrum of $T$. Show that there exists a sequence $x_n \in E$, $\|x_n\| = 1$, such that $Tx_n - \lambda x_n \to 0$, as $n \to \infty$.

6. Show that the spectrum $\sigma(T)$ of $T \in B(E)$, $E$ is a Banach space, coincides with the spectrum $\sigma(T^*)$ of the adjoint operator $T^* : E^* \to E^*$. Moreover,

   - if $\lambda \in \sigma_r(T)$, then $\lambda \in \sigma_p(T^*)$;
   - if $\lambda \in \sigma_p(T)$, then either $\lambda \in \sigma_p(T^*)$ or $\lambda \in \sigma_r(T^*)$;
   - if $\lambda \in \sigma_c(T)$, then either $\lambda \in \sigma_c(T^*)$ or $\lambda \in \sigma_r(T^*)$; if the space if reflexive the second possibility is impossible.

7. Let $A$ be a diagonal operator on $\ell^p$, $1 \leq p \leq \infty$, given by

$$A(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots),$$

   where $(a_i)_i$ is a bounded sequence. Show that $\sigma(A)$ is the closure of $\{a_i : i \in \mathbb{N}\}$, $\sigma_p(A) = \{a_i : i \in \mathbb{N}\}$. Moreover, $\sigma(A) \setminus \sigma_p(A)$ is $\sigma_c(A)$ if $p < \infty$ and $\sigma_r(A)$ if $p = \infty$.

8. Let $(a_n)$ be a bounded sequence of complex numbers and let $H$ be a Hilbert space having an orthonormal basis $\{e_i\}$.

   - Show that there is a (necessarily unique) bounded operator $A \in B(H)$ satisfying $Ae_n = a_n e_{n+1}$, $n = 1, 2, \ldots$. Such an operator $A$ is called a unilateral weighted shift.
   - Show that for every complex number $\lambda$, $|\lambda|$ there is a unitary operator $U = U_\lambda$ such that $UAU^{-1} = \lambda A$.
   - Deduce that the spectrum of a weighted shift must be the union of (possibly degenerate) concentric circles about $z = 0$.

9. Let $B$ be the left shift operator as defined in Exercise 2 on $\ell^p$, $1 \leq p \leq \infty$. Determine the four sets $\rho(B)$, $\sigma_p(B)$, $\sigma_c(B)$, $\sigma_r(B)$ for all values of $p$. Hint: Exercise 6 and a modification of 8 can be useful.

Exercises 1-4 and 8 are from Arveson " A short course on spectral theory"