Exercises in Spectral Theory and Operator Algebras II

1. Consider $L^1(\mathbb{R})$ and define a multiplication on $L^1(\mathbb{R})$ by convolution

$$f * g(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt, f, g \in L^1(\mathbb{R}).$$

   $(L^1(\mathbb{R})), *)$ is a commutative Banach algebra.

   - Prove that $(L^1(\mathbb{R}), *)$ does not have an identity.
   - For every $n = 1, 2, \ldots$, let ϕ_n be a nonnegative function in $L^1(\mathbb{R})$ such that ϕ_n vanishes outside $[-1/n, 1/n]$ and $\|\phi_n\|_1 = 1$. Show that $\{\phi_n\}_n$ is an approximate identity in the sense that $\|f * \phi_n - f\|_1 \to 0$ as $n \to \infty$, for all $f \in L^1(\mathbb{R})$.
   - Let $f \in L^1(\mathbb{R})$. The Fourier transform of f is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{it\xi} f(t)dt, \xi \in \mathbb{R}.$$

     Show that $+hatf \in C_0(\mathbb{R})$, the algebra of all continuous functions vanishing at infinity.
   - Show that the Fourier transform is a homomorphism of $(L^1(\mathbb{R})), *)$ onto a subalgebra of $C_0(\mathbb{R})$ which is closed under complex conjugation and separates points of $\mathbb{R}$.

2. An operator $A \in B(E)$ is called compact if the norm closure of the image of the unit ball in E under A , i.e. $\{Ax : x \in E, \|x\| = 1\}$ is a compact subset of E. A is called finite-dimensional if the dimension of the vector space AE is finite.

   - Let $A \in B(E)$ be an operator with the property that there is a sequence of finite rank operators $\{A_n\}_n$ such that $\|A - A_n\| \to 0$ as $n \to \infty$. Show that A is compact.
   - Let $\{a_n\}_n$ is a bounded sequence of complex numbers and let A be a diagonal operator on the Hilbert space ℓ^2,

$$A\xi(n) = a_n \xi(n), \xi \in \ell^2, n \in \mathbb{N}.$$

     Show that A is compact iff $\lim_{n\to\infty} a_n \to 0$.

3. Let k be a continuous complex-valued function defined on $[0, 1] \times [0, 1]$. Show that the operator A defined on $C[0, 1]$ by

$$(Af)(x) = \int_0^1 k(x, y)f(y)dy, x \in [0, 1]$$

is a compact linear operator on $C[0,1]$. Hint: Start to looking at the case $k(x,y) = u(x)v(y)$ with u,$v \in C[0,1]$ and apply Exercise 2.

4. Let T be the operator defined on $L^2([0,1])$ by $Tf(x) = xf(x)$, $x \in [0,1]$. What is the spectrum of T? Does T have point spectrum?

5. Let $\{a_i\}$ be a sequence of complex numbers such that an $a_n \to 0$. Show that the associated weighted shift operator on ℓ^2 given by

$$Ae_n = a_n e_{n+1}, n \geq 1,$$

has spectrum $\{0\}$.

6. Show that every normed linear space E has a basis $B \subset E$ consisting of unit vectors, and deduce that every infinite-dimensional normed linear space has a discontinuous linear functional $l : E \to \mathbb{C}$. Recall that a basis for a vector space V is a set of vectors B with the following two properties: every finite subset of B is linearly independent, and every vector in V is a finite linear combination of elements of B.

7. Let A be a complex algebra and let I be a proper ideal of A. Show that I is a maximal ideal iff the quotient algebra A/I is simple.

8. Let A be a unital Banach algebra, let n be a positive integer, and let $w : A \to M_n$ be a homomorphism of complex algebras such that $w(A) = M_n$, M_n denoting the algebra of all $n \times n$ matrices over $\mathbb{C}$. Show that w is continuous (where M_n is topologized in the natural way by $\mathbb{C}^{n^2}$. Deduce that every linear functional $f : A \to \mathbb{C}$ satisfying $f(xy) = f(x)f(y)$, $x, y \in A$, is continuous.

9. Let A be the algebra of continuous functions on the complex unit circle $\mathbb{T}$, and B the subalgebra of functions which are restriction to $\mathbb{T}$ of functions which are analytic on the open complex unit disc $\mathbb{D}$ and continuous on the closed complex unit disc $\bar{\mathbb{D}}$. Show that for $f(z) = z$ we have $\sigma_A(f) = \mathbb{T}$ but $\sigma_B(f) = \bar{\mathbb{D}}$, where $\sigma_A(f)$ ($\sigma_B(f)$) denotes the spectrum of f in A (resp. in B).

10. Let $a \in A$ be an element of the Banach algebra A; and let $\mathcal{O} \subset \mathbb{C}$, $\sigma(a) \subset \mathcal{O}$. Show that there exists $\delta > 0$ such that $\sigma(b) \subset O$ for any $b \in A$ such that $\|b - a\| \leq \delta$. Hint: use the fact that the set of invertible elements in Banach algebra is open.

11.
    - Show that the Banach spaces ℓ^∞ and $c = \{\{x_n\} \in \ell^\infty : \exists \lim_n x_n\}$ provided with pointwise multiplication are commutative Banach algebras.
    - Show that $c_0 = \{\{x_n\} \in \ell^\infty : \lim_n x_n = 0\}$ is a (norm) closed proper ideal in ℓ^∞.
    - Show that c_0 is a maximal ideal in c finding a linear multiplicative functional with c_0 as its kernel.
    - Is c_0 a maximal ideal in ℓ^∞?

12. Let V be the Volterra operator on $L^2[0,1]$ and let A be the Banach subalgebra of $B(L^2[0,1])$ generated by V and the identity operator, i.e. A is the smallest Banach algebra with identity that contains V. Determine the maximal ideal space of A and the radical of A.

Exercises 1-8 are from Arveson " A short course on spectral theory"