Exercises in Spectral Theory and Operator Algebras III

In Exercises 1-4, A is a commutative Banach algebra with unit.

1. Show that if A is nontrivial in the sense that $A \neq \{0\}$, one has $\mathrm{sp}(A) \neq \emptyset$.

2. Show that the mapping $\omega \in \mathrm{sp}(A) \to \ker \omega$ is a bijection of the Gelfand spectrum onto the set of all maximal ideals in A.

3. Show that the Gelfand map is an isometry, i.e. $\|\Gamma(a)\| = \|a\|$ for all $a \in A$, iff $\|a^2\| = \|a\|^2$ for every $a \in A$.

4. The radical of A is defined as the set $\mathrm{rad}(A)$ of all quasinilpotent elements of A. Show that $\mathrm{rad}(A)$ is a closed ideal in A with the property that $A/\mathrm{rad}(A)$ has no quasinilpotent elements (such A is called semisimple).

5. Let A and B be commutative unital Banach algebras and let $\theta : A \to B$ be a homomorphism of the complex algebra structures such that $\theta(1_A) = 1_B$. Do not assume that θ is continuous.

   - Show that θ induces a continuous map $\hat{\theta} : \mathrm{sp}(B) \to \mathrm{sp}(A)$ by $\hat{\theta}(\omega) = \omega \circ \theta$.
   - Assuming that B is semisimple, show that θ is necessarily bounded. Hint: Use the closed graph theorem.
   - Deduce that every automorphism of a commutative semisimple unital Banach algebra is a topological automorphism, i.e. continuous.

6. Let $A = C^1([0,1])$. A is a Banach algebra with respect to the norm $\|f\| = \sup_{[0,1]} |f(t)| + \sup_{[0,1]} |f'(t)|$. Consider $x : [0,1] \to \mathbb{C}$, $x(t) = t$. Show that x generates A as a Banach algebra, i.e. A is the smallest Banach algebra with unit that contains x. Then show that $[0,1] \to \mathrm{sp}(A)$, $t \to \omega_t$, where $\omega_t(f) = f(t)$ is a homeomorphism and that the spectral radius $r(f) = \|f\|_\infty$, $f \in C^1([0,1])$. Using this prove then that the Gelfand transform is not surjective.

7. Let $[\cdot,\cdot] : H \times H \to \mathbb{C}$ be a sesquilinear form defined on a Hilbert space H. Show that $[\cdot,\cdot]$ satisfies the polarization formula
$$4[\xi,\eta] = \sum_{k=0}^{3} [\xi + i^k\eta, \xi + i^k\eta].$$

8. Let $A \in B(H)$ be a Hilbert space operator. The quadratic form of A is the function $q_A : H \to \mathbb{C}$ defined by $q_A(\xi) = \langle A\xi, \xi\rangle$. The numerical range and numerical radius of A are defined, respectively, by
$$W(A) = \{q_A(\xi) : \|\xi\| = 1\} \subset \mathbb{C}$$
$$\omega(A) = \sup\{|q_A(\xi)| : \|\xi\| = 1\}.$$

- Show that A is selfadjoint iff q_A is real-valued.
- Show that $\omega(A) \leq \|A\| \leq 2\omega(A)$ and deduce that $q_A = q_B$ only when $A = B$. Hint: Polarize.
- Show that if A is selfadjoint then $q_{A^2}(\xi) = q_A(\xi)^2$, $\|\xi\| = 1$, only if ξ is an eigenvector. Hint: Use Cauchy-Schwartz.

Exercises 1-5, 7,8 are from Arveson " A short course on spectral theory"