Exercises in Spectral Theory and Operator Algebras IV

C^*-algebras.

1. Let A be a C^*-algebra.

   (a) Show that the involution in A satisfies $\|a^*\| = \|a\|$, $a \in A$.

   (b) Show that if A contains a unit $\mathbf{1}$, then $\|\mathbf{1}\| = 1$.

   In the next three exercises, X and Y denote compact Hausdorff spaces and $\theta : C(X) \to C(Y)$ denotes an isomorphism of complex algebras. Continuity is not assumed.

2. Let $p \in Y$. Show that there is a unique point $q \in X$ such that

$$\theta f(p) = f(q), f \in C(X).$$

3. Show that there is a homeomorphism $\phi : Y \to X$ such that $\theta f = f \circ \phi$. Hint: Think in terms of the Gelfand spectrum.

4. Conclude that θ is necessarily a self-adjoint linear map in the sense that $\theta(f^*) = \theta(f^*)$, $f \in C(X)$.

5. Formulate and prove a theorem that characterizes unital algebra homomorphisms $\theta : C(X) \to C(Y)$ in terms of certain maps $\phi : Y \to X$. Which maps give rise to isomorphisms?

   Next exercise shows that for non-commutative C^*-algebras we may have isomorphisms that do not preserve involution.

6. Let H be a Hilbert space and let $T \in B(H)$ be an invertible bounded operator. Define $\theta : B(H) \to B(H)$ by $\theta(A) = TAT^{-1}$, $A \in B(H)$. Show that θ is an automorphism of the Banach algebra structure of $B(H)$ and that $\theta(A^*) = \theta(A)^*$ for all $A \in B(H)$ if and only if T is a multiple of a unitary operator.

   Functional calculus.

7. Show that the spectrum of a normal operator $T \in B(H)$ is connected if and only if the C^*-algebra generated by T and $\mathbf{1}$ contains no projections other than $\mathbf{0}$ and $\mathbf{1}$.

   Consider the algebra $\mathcal{C}$ of all continuous functions $f : \mathbb{C} \to \mathbb{C}$. There is no natural norm on $\mathcal{C}$, but for every compact $X \subset \mathbb{C}$ there is a seminorm

$$\|f\|_X = \sup_{z \in X} |f(z)|.$$

   $\mathcal{C}$ is a commutative $*$-algebra with unit.

8. Given a normal operator $T \in B(H)$, show that there is a natural extension of the functional calculus to a $*$-homomorphism $f \in \mathcal{C} \to f(T) \in B(H)$ that satisfies $\|f(T)\| = \|f\|_{C(\sigma(T))}$.

   Fix a function $f \in \mathcal{C}$ and let $(T_n)_n$ be a sequence of normal operators that converges in norm to an operator T. Show that $(f(T_n))_n$ converges to $f(T)$.

   Diagonalization.

9. Let X be a Borel space, let f be a bounded complex-valued Borel function defined on X, and let μ and ν be two σ-finite measures on X. The multiplication operator M_f defines a bounded operator A on $L^2(X,\mu)$ and B on $L^2(Y,\nu)$. Assuming that μ and ν are mutually absolutely continuous, show that there is a unitary operator $W : L^2(X,\mu) \to L^2(Y,\nu)$ such that $WA = BW$. Hint: Use the Radon-Nikodym theorem.

10. Show taht every diagonalizable operator on a separable Hilbert space is unitarily equivalent to a multiplication operator M_f acting on $L^2(X,\mu)$, where (X,μ) is a probability measure space, i.e. $\mu(X) = 1$.

11. The following exercise concerns the selfadjoint operator A defined on $\ell^2(\mathbb{Z})$ by

$$(A\xi)_n = \xi_{n+1} + \xi_{n-1}, n \in \mathbb{Z}, \xi = (\xi_n)_{n\in\mathbb{Z}}.$$

   (a) Show that A is diagonalizable by exhibiting an explicit unitary operator $W : L^2(\mathbb{T}, d\theta/2\pi) \to \ell^2(\mathbb{Z})$ for which $WM_f = AW$, where $f : \mathbb{T} \to \mathbb{R}$ is $f(e^{i\theta}) = 2\cos\theta$. What is the spectrum/the point spectrum of A?

   (b) Let U be a unitary operator defined on $L^2(\mathbb{T}, d\theta/2\pi)$ by

$$U\xi(e^{i\theta}) = \xi(e^{-i\theta}), 0 \leq \theta \leq 2\pi$$

   Show that U is a unitary operator $U^2 = \mathbf{1}$, and which commutes with M_f with f given above.

   (c) Let $\mathcal{B}$ be the set of all operators $L^2(\mathbb{T}, d\theta/2\pi)$ that have the form $M_g + UM_h$, where g, $h \in L^\infty(\mathbb{T}, d\theta/2\pi)$ and U is the unitary given in the previous paragraph. Show that $\mathcal{B}$ is a C^*-algebra isomorphic to the C^*-algebra of all 2×2 matrices of functions $M_2(\mathcal{B}_0)$, where $\mathcal{B}_0$ is the abelian C^*-algebra $L^\infty(\mathbb{T}^+, d\theta/2\pi)$, $\mathbb{T}^+$ being the upper half of the unit circle $\mathbb{T}$.

12. This exercise asks you to compare A to a related operator B on $L^2([-2,2],\nu)$, ν being Lebesque measure on $[-2,2]$, given by

$$B\xi(x) = x\xi(x), \xi \in L^2([-2,2],\nu).$$

   (a) Argue that B has spectrum $[-2,2]$, no point spectrum and deduce that $\|f(A)\| = \|f(B)\|$ for every $f \in C[-2,2]$.

   (b) Show that A and B are not unitarily equivalent. Hint: Compare the commutants of A and B that is the algebras of operators commuting with A and B respectively.

   (c) Show that A is unitarily equivalent to $B \oplus B$.

Assignment: 11 a,b and 12 a,b. You may also try to do 11c, it can be used to answer 12b.

The exercises are from Arveson " A short course on spectral theory"