Exercises in Spectral Theory and Operator Algebras V

1. Let r be a real number satisfying $0 < r < 1$. Show that an infinite-dimensional Banach space contains a sequence of unit vectors $\{u_n\}$ satisfying $\|u_n - u_m\| \geq r$ for all $n \neq m$. Deduce then that the unit ball of a Banach space E is compact iff E is finite-dimensional.

2. Let F be a finite-dimensional subspace of a Banach space E. Show that there is an operator $P \in B(E)$ satisfying $P^2 = P$ and $PE = F$. Hint: Pick a basis $x_1, \dots, x_n$ for F and find bounded linear functionals $f_1, \dots, f_n$ on E such that $f_i(x_j) = \delta_{ij}$. Deduce then that every finite-dimensional subspace $F \subset E$ is complemented in the sense that there is a closed subspace $G \subset E$ with $G \cap F = \{0\}$ and $G + F = E$.

3. Show that for any Banach space E, $\mathcal{K}(E)$ is a norm-closed two-sided ideal in $B(E)$.

4. Let $T \in \mathcal{K}(E)$. Show that $T^* \in \mathcal{K}(E^*)$. Hint: Use Ascoli's theorem.

5. Find a necessary and sufficient condition that the diagonal operator T on ℓ^p, $1 \leq p < \infty$, given by
$$T(x_1, x_2, \dots) = (a_1 x_1, a_2 x_2, \dots)$$
is compact.

6. Show that $M_f : L^2([a,b]) \to L^2([a,b])$, $M_f \xi(x) = f(x)\xi(x)$, $f \in L^\infty([a,b])$ is compact iff $f = 0$.

7. Let $C_2(H)$ be the space of Hilbert-Schmidt operator on a separable Hilbert space H. Let $\{e_k\}$ be an orthonormal basis in H. Show that $\langle A, B\rangle = \sum_k \langle A_k, Be_k\rangle$ defines an inner product on $C_2(H)$ and $C_2(H)$ is complete with respect to $\|A\|_2 = \langle A, A\rangle^{1/2}$. Show also that $\|A\| \leq \|A\|_2$, $A \in C_2(H)$. Moreover, if $k \in L^2(X \times X, \mu \times \mu)$, then $\|I_k\|_2 = \|k\|_2$, where $I_k \in B(L^2(X, \mu))$ is given by
$$I_k \xi(x) = \int k(x,y)\xi(y)d\mu(y), \xi \in L^2(X, \mu).$$

8. Let (X, μ) be a finite measure space. We will call a system of measurable subsets $D = \{\alpha_j \times \beta_j \subset X \times X : 1 \leq j \leq J\}$ diagonal if $\alpha_i \cap \alpha_j = \emptyset$, $\beta_i \cap \beta_j = \emptyset$. For such D let
$$\pi_D(T) = \sum_{j=1}^{J} M_{\chi_{\beta_j}} T M_{\chi_{\alpha_j}}, T \in B(L^2(X, \mu)),$$
and $r(D) = \max_{1 \leq j \leq J}(\min\{\mu(\alpha_j), \mu(\beta_j)\})$. Show that if D^k is a sequence of diagonal systems such that $r(D^k) \to 0$, $k \to \infty$. Then $\|\pi_{D^k}(T)\| \to 0$ for each compact operator T. Hint: prove the statement first for rank one operators using the previous exercise.

9. Can a compact operator A on infinite-dimensional space satisfy the equation $\sum_{k=0}^{n} c_k A^k = 0$, where we set $A^0 = I$. Formulate a criterium when this is possible.

10. Find the spectrum of the compact operator on $L^2([0,1])$ given by

$$Ax(t) = \int_0^1 \min\{t, s\} x(s) ds, x \in L^2([0,1]).$$

11. Find parameters p, q, r for which the integral equation

$$x(t) = \lambda \int_{-1}^{1} (1 + 2t + 3st) x(s) ds + pt^2 + qt + r$$

has a solution in $L^2([-1,1])$ for any $\lambda \in \mathbb{C}$.

Assignment: 8, 10, 11.
The exercises 1-4 are from Arveson " A short course on spectral theory"