Exercises in Spectral Theory and Operator Algebras VI

Let E be an infinite-dimensional space.

1. (a) Show that a Fredholm operator T on E is a compact perturbation of an invertible operator iff its index vanishes. Hint: If $\text{ind}T = 0$, show how to construct a finite rank perturbation of T that is one-to-one and onto.

   (b) Deduce the following concrete description of the equivalence relation $A \sim B \Leftrightarrow \text{ind}A = \text{ind}B$: Two Fredholm operators A and B on E have the same index iff there is an invertible C such that $A - BC$ is compact.

2. Let S be the unilateral shift acting on a Hilbert space H ($Se_n = e_{n+1}$, $\{e_n\}$ is an orthonormal basis in H).

   (a) Show that there is no compact operator K such that $S + K$ is invertible.

   (b) Let T be a Fredholm operator of positive index n. Show that there is an invertible operator C such that and a compact operator K such that $T = (S^*)^n C + K$.

3. (a) Let N be a normal Fredholm operator on a Hilbert space H. Show that the index of N vanishes.

   (b) Deduce that the unilateral shift S is not a compact perturbation of a normal operator.

4. With S as in the preceding exercises, let $S \oplus S^* \in B(H \oplus H)$ be the direct sum of S with its adjoint S^*. Show that $S \oplus S^*$ is a Fredholm operator and calculate its index.

5. Let U be the bilateral shift, defined on a Hilbert space H by its action on a bilateral orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ for H by $Ue_n = e_{n+1}$, $n \in \mathbb{Z}$. Let P be the projection onto the one dimensional space spanned by e_0. Show that $U - UP$ is unitarily equivalent to $S \oplus S^*$ of the proceeding exercise, and deduce that $S \oplus S^*$ is a compact perturbation of a normal operator.

6. Let $\{a_n\}$ be a bounded sequence and let $A : \ell^p \to \ell^p$, $1 \leq p < \infty$ be the diagonal operator given by $Ae_n = a_n e_n$. Show that A is Freholm iff 0 is not an accumulating point of the sequence $\{a_n\}$, and if it is Fredholm then its index is zero.

7. Let $M_f : L^p([0,1]) \to L^p([0,1])$, $1 \leq p < \infty$, where $f \in L^\infty([0,1])$, be the multiplication operator. Show that M_f is Fredholm iff M_f is invertible.

The exercises 1-5 are from Arveson " A short course on spectral theory"