Questions to the oral exam in Spectral Theory and Operator Algebras, 2019

1. Spectrum of an Operator. Type of Spectrum. Examples.
2. Banach and C^*-algebras: definitions and examples. Spectrum of an element of a Banach algebra: general properties and Gelfand theorem about spectrum.
3. Spectral radius and quasi-nilpotent elements. Selfadjoint elements of a C^*-algebra and quasinilpotency.
4. Commutative Banach algebras. The Gelfand map, spectrum as the range of the Gelfand transform, connection with maximal ideals.
5. The Gelfand map: injectivity/surjectivity of the map. Examples. The spectrum of $C(X)$, spectrum of $C^*(a, 1)$, where a is a normal element of a unital C^*-algebra and $C^*(a, 1)$ is the C^*-subalgebra generated by a and 1.
6. Operators on Hilbert spaces. Adjoint operator.
7. Spectrum of the multiplication operator, essential range.
8. Gelfand theory of commutative C^*-algebras.
9. Continuous functional calculus for normal operators: construction of the $*$-isomorphism $f \mapsto f(A)$ from $C(\sigma(A))$ to $C^*(A, 1)$, where A is a normal operator and $C^*(A, 1)$ is the unital C^*-algebra generated by A and 1.
10. Diagonalization: definition, Spectral theorem.
11. Compact operators: definition, examples of compact and non-compact operators, compactness of the identity operator, diagonal operators on ℓ^2, Hilbert-Schmidt operators.
12. Spectral theorem for normal compact operators on separable Hilbert spaces.
13. Riesz theory of compact operators on general Banach spaces: Fredholm alternative and countability of spectrum.
14. Fredholm operators and index. Examples of Fredholm and non-Fredholm operators. Atkinson's theorem and main propertirs of index.
15. Toeplitz operators. Definition. When such operators are compact, Fredholm? Index of a Fredholm Toeplitz operator with continuous symbol.

The list of theorems that you should be able to explain the proofs.

1. Theorem about invertibility of an element $1-x \in A$, (A - unital Banach algebra) with $\|x\| < 1$. Show that the set of invertible elements is open (Theorem 1.5.2, Corollary 1).

2. Gelfand theorem about spectrum of an element of a Banach algebra (Proposition 1.6.2 and Theorem 1.6.3)

3. Gelfand-Mazur-Beurling theorem about spectral radius ($r(x) = \lim_n \|x^n\|^{1/n}$) (Theorem 1.6.3)

4. Gelfand theorem about spectrum of an element of a commutative unital Banach algebra as image of the Gelfand trasform of the element (Theorem 1.9.5)

5. Gelfand theorem about the structure of commutative C^*-algebras (Theorem 2.2.4)

6. Spectral theorem for normal operators on separable Hilbert space: Know main steps of the proof (Theorem 2.4.5)

7. Theorem about countability of spectrum of a compact operator (Theorem 3.2.3). Deduce spectral theorem for normal compact operators on Hilbert spaces (without use of functional calculus for normal operators).

8. Atkinson's theorem (Theorem 3.3.2) and statements about stability of index and its continuity (Corollary 2, 3).

9. Show that every Toeplitz operator T_ϕ, $\phi \in L^\infty$ satisfies $\inf\{\|T_\phi + K\| : K \in \mathcal{K}\} = \|T_\phi\| = \|\phi\|_\infty$ (Corollary 1 to Theorem 4.2.4)

10. Show that the Toeplitz C^*-algebra consists of all operators of the form $T_f + K$, where $f \in C(\mathbb{T})$, $K \in \mathcal{K}$. (Theorem 4.3.2) Derive a corollary about Fredholm Toeplitz operators with continuous symbol (Corollary 1).