Exercises

September 2, 2011
due
September 9, 2011

1 Show that the Moebius transformations $z, \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{z}{z-1}, \frac{z-1}{z}$ are closed under composition and make up a group isomorphic with $S_3(D_6)$. In particular identify the involutions, i.e. elements of order two.

Hint: *Show that the group permutes the elements* $0, 1, \infty$

2 Consider the invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ and the entries $a, b, c, d \in \mathbb{Z}_2$. Show that those make up a group under multiplication, and which is also isomorphic with the symmetric group S_3. Identify the elements of order two, and show that they are characterized by their traces.

Hint: *Show that the group naturally acts on the vector space* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and permutes the non-zero elements*

3 Find all the subgroups of S_4 (the symmetric group on four elements).

4 Find the number of conjugacy classes of S_7 and find explicitly a Sylow 2-group.

5 If a, b are arbitrary elements in a group G, show that ab and ba are conjugate, in particular that they have the same order.

6 Consider elements a, b that satisfy $a^2 = b^3 = 1$ and consider all the (reduced) words made up of those letters. Thus aba, ab^2ab are examples of reduced words, while $abaa$ is not reduced but can be simplified to ab.

a) Show that the set of all reduced words make up a group. It is the so called free product of the cyclic groups of order two and three respectively. It is called the Modular group Γ and plays an important part in mathematics.

b) An element has a natural length, the identity has length zero (the empty word) while it is convenient to think of $b^2 = b^{-1}$ as length one. Thus ab^2 has length two while aba has length three. Can you find a closed (recursive) formula for the number of words of length n?

c) Show that we can define a group homomorphism $\Gamma \to S_3$ by assigning a to an involution and b to an element of order 3. Show that the kernel is generated freely by the elements $X = abab$ and $Y = baba$. I.e. both X and Y are elements of infinite order, and that all the words $X^{n_1}Y^{m_1}X^{n_2}\dots$ with $n_i, m_i \in \mathbb{Z}(n_i, m_i \neq 0)$ are distinct.

Hint: *Consider words of* X, Y *as words in* a, b *and that the lengths of the latter strictly increase when you compose words in* X, Y