Exercises

September 30, 2011
due
October 7, 2011

1 Consider the infinite direct product of a field k.
a) Show that it has an uncountable dimension as a vector space over k
b) Try to find an 'explicit' example of an uncountable set of linearly independent elements!

2 Let N be a Noetherian module, and let $\phi : N \to N \to 0$ be a surjection. Show that ϕ is in fact a bijection.

3 Find an example of a surjective map $M \to \mathbb{Q} \to 0$ of $\mathbb{Z}$-modules which does not split.

4 Is it possible to find an injection $0 \to \mathbb{Q} \to F$ where F is a free module?

5 Let Λ be the sub-lattice of $\mathbb{Z}^2$ generated by $(6,2),(2,2)$ find the decomposition of the torsion module $\mathbb{Z}^2/\Lambda$.

6 In how many different ways can $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ be decomposed into a sum of cyclic subgroups $\mathbb{Z}/2\mathbb{Z}$?

7 A finite abelian group A is given by the elementary factors $2|10|490$
a) Compute the order of A!
b) Identify the elementary factors of $\mathrm{End}(A)$!

8 How many different abelian groups are there of order $6!(=720)$

9 Show that $p(x) = x^3 + 3x + 1$ is an irreducible element of $\mathbb{Q}[x]$ and give an example of non-zero matrix $A \in M_3(\mathbb{Q})$ such that $p(A) = 0$.

Hint: *If not irreducible $p(X)$ has a linear factor, hence a rational root. Show that this is impossible.* Note that the 3-dimensional vectorspace $\mathbb{Q}[x]/(x^3+3x+1)$ is an irreducible $\mathbb{Q}[x]$-module.