Representations of $\mathfrak{sl}(2,\mathbb{C})$

Let us consider the standard generators X, Y, H of $\mathfrak{sl}(2,\mathbb{C})$ satisfying the commutator relationships

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

Now let V be a finite-dimensional irreducible representation. Under the action of H we can split up V into eigenspaces $V = \oplus_\alpha V_\alpha$ with a finite number of eigenvalues α.

The first basic fact we want to establish is that X and Y permute these eigenspaces, which follows from two simple calculations.

So let $v \in V_\alpha$

$$HXv = XHv + [H, X]v = \alpha Xv + 2Xv = (\alpha + 2)Xv$$
$$HYv = YHv + [H, Y]v = \alpha Yv - 2Yv = (\alpha - 2)Yv$$

Thus X shifts "two steps" to the "right" $XV_\alpha \subset V_{\alpha+2}$ while Y shifts to the left $YV_\alpha \subset V_{\alpha-2}$

In particular XY and YX (none of which actually belong to $\mathfrak{sl}(2)$ operate on each of the eigenspaces V_α. The underlying idea is now to look at eigenvectors of these actions.

More specifically, consider $V_\alpha \neq 0$ such that $XV_\alpha = 0$. This is always possible for any non-trivial finite-dimensional V.

Pick a non-trivial element $v \in V_\alpha$. The contention is that it is automatically an eigenvector for XY (and trivially YX). In fact

$$XYv = YXv + [X, Y]v = HV = \alpha v$$

Hence its eigenvalue is α.

We would now like to look at the iterates $v, Yv, Y^2v \dots$. They are all linearly independent as they belong to distinct eigenvalues. But they are also all eigenvectors for XY. In fact by induction we can assume that $XY(Y^k v) = \lambda(k)Y^k v$ with $\lambda(0) = \alpha$. Thus

$$XY(Y^{k+1}v) = YX(Y^{k+1}v) + HY^{k+1}v = YXY(Y^k v) + (\alpha - 2(k+1))v =$$

$$= (\lambda(k) + \alpha - 2(k+1))Y^{k+1}$$

showing incidentally that $\lambda(k+1) = \lambda(k) + \alpha - 2(k+1)$. By induction we prove that $\lambda(k) = (k+1)\alpha - k(k+1)$ But the fact that $Y^k v$ are eigenvectors for XY means that X permutes, in fact shifts, the iterates $Y^k v$.

$$XY^{k+1}v = \lambda(k)Y^k v$$

. Thus the linear space spanned by those iterates is not only invariant under Y (by construction) and H (each iterate an eigenvector of H) but also under X. By irreducibility we conclude that they span the entire space V.

Finally the iterates can only be finite in number, there is an $N > 0$ such that $Y^{N+1}v = 0$ but $Y^N v \neq 0$. This means

$$0 = XY^{N+1}v = \lambda(N+1)Y^N v$$

thus $\lambda(N+1) = 0$ which translates into $\alpha = N$.

We have now elucidated the structure of the finite-dimensional irreducible representations of $\mathfrak{sl}(2)$. The pertinent facts are

i) The only eigenvalues are integers, and they all differ by multiples of two.

ii) All eigenvalues belong to 1-dimensional eigenspaces.

The top eigenvalue is N and $Y^k v$ correspond to $N - 2k$. In particular the lowest eigenvalue will be $-N$. Thus in particular we see that the eigenvalues are symmetrical with respect to the origin.

All irreducible representations corresponding to a given N are isomorphic, and by using the basis $e_i = Y^k v$ with $i = (N - 2k)$ we can describe the representation as follows

$$\begin{aligned} He_i &= ie_i \\ Ye_i &= e_{i-2} \\ Xe_i &= \frac{(N+1)^2 - (i+1)^2}{4} e_{i+2} \end{aligned}$$

for a set of vectors $e_{-N}, e_{-N+2}, \dots e_{N-2}, e_N$ with an obvious convention for indices going out of bound. Note that such a representation is $N+1$ dimensional

It is also straightforward to check that this does indeed give a representation of $\mathfrak{sl}(2)$ by checking the commutator relations.

So let us look at the cases for small N.

$N = 0$ This is the trivial one dimensioanl representation

$N = 1$ This is the canonical, or standard, representation of $\mathfrak{sl}(2)$ on $\mathbb{C}^2$ as a lie subalgebra of $\mathfrak{gl}(2, \mathbb{C})$ The matrix representation fo H, X and Y becomes the standard.

$N = 2$ This is the adjoint representation of $\mathfrak{sl}(2)$ on itself. The irreducibility of it shows that $\mathfrak{sl}(2)$ is simple. The eigenspaces of H corresponding to $-2, 0, 2$ are of course spanned by Y, H and X.

Now any (finite-dimensional) representation of $\mathfrak{sl}(2)$ is a sum of irreducible. From this we can conclude som general facts.

So let V be a representation and let $w(\alpha) = \dim V_\alpha$. Then the weight function w takes non-zero values only on the integers, and is zero for almost all.

Furthermore

A) $w(\alpha) = w(-\alpha)$

B) V has a unique decomposition $V = V_+ \oplus V_-$ such that the corresponding weight functions w_- and w_+ vanish on even and odd integers respectively. For those weight functions we have uni-modularity i.e $w(\alpha) \geq w(\beta)$ if $|\alpha| \leq |\beta|$.

We also note that we can recover the representation from the weight function, thus it determines the representation uniquely. Furthermore any function satisfying A) and B) can occur as a weight-function.

The symmetry condition A) shows that the dual V^* of a $\mathfrak{sl}(2)$ representation is isomorphic to V. Another way of putting this is to say that there is a $\mathfrak{sl}(2)$ invariant linear map $q : V \to V^*$ which gives rise to a $\mathfrak{sl}(2)$ invariant non-degenerate bilinear form Q via $Q(x, y) := q(x)(y)$. Conversely given such a bilinear form, we can get an invariant map $q : V \to V^*$ by reversing the equality $q(x)(y) := Q(x, y)$. We have hence showed that the space $V \times V$ contains trivial summands (intersecting the open subset non-degenerate forms).

In general we would like to compute the weight function for representations. So let us consider some standard examples.

$V \otimes W$. Writing $V = \oplus V_\alpha$ and $W = \oplus W_\beta$ we get a decomposition $V \otimes W = \oplus V_\alpha \otimes W_\beta$. If $v \otimes w \in V_\alpha \otimes W_\beta$ then $H(v \otimes w) = Hv \otimes w + v \otimes Hw = (\alpha + \beta) v \otimes w$. Hence $V_\alpha \otimes W_\beta$ corresponds to eigenvectors with eigenvalue $\alpha + \beta$.

As we have $\dim V = \sum_\alpha \dim V_\alpha$ and $\dim W = \sum_\beta \dim W_\beta$ we get

$$\dim V \otimes W = \sum_{\alpha,\beta} \dim V_\alpha \dim W_\beta = \sum_{\alpha,\beta} \dim V_\alpha \otimes W_\beta$$

exhausting the tensor product.

Hence we can write down the weightfunction $w_{V \otimes W}$ in terms of the weightfunctions w_V and w_W. In fact we get

$$w_{V \otimes W}(\gamma) = \sum_{\alpha+\beta} w_V(\alpha) w_W(\beta)$$

thus in fact it is given by the convolution of w_V and w_W.

In particular this applies to $\mathrm{Hom}(V, W) = V^* \otimes W = V \otimes W$. We can also look at $V = W$ and the symmetric and alternating part.

Clearly

$$w_{S^2(V)}(\gamma) = \sum_{\alpha \leq \beta} w_V(\alpha) w_V(\beta)$$

$$w_{\bigwedge^2(V)}(\gamma) = \sum_{\alpha < \beta} w_V(\alpha) w_V(\beta)$$

More generally we can write down the corresponding formulas for $S^n(V)$ and $\bigwedge^n(V)$.

If V is the standard 2-dimensional representation of $\mathfrak{sl}(2)$ given by $x \in V_1$ and $y \in V_{-1}$ we can consider $S^n(V)$ the vectorspace of all binary forms of degree n.

By the product rule we easily see that $Hx^n = nx^n$ and $Hy^n = -ny^n$ and more generally $Hx^k y^{n-k} = (n - 2k) x^k y^{n-k}$. Thus we recognise this as the unique irreducible module of dimension $n + 1$. Thus from now on we will have a standard notation for them.

Thus a general problem (an equivalent reformulation of the weightfunction) is to decompose any representation into sums of the $S^n(V)$. So let us do this for a few examples.

So let V be the standard representation with $w(1) = w(-1) = 1$. Then the weight function of $V \times V$ will satisfy $w(-2) = w(2) = 1, w(0) = 2$ hence we have the decomposition

$$\mathrm{Hom}(V, V) = V \times V = S^2(V) \oplus \mathbb{C}$$

Where the last summand is the trivial representation, which is actually isomorphic with $\bigwedge^2(V)$.

Now we have in general a map $\mathrm{Hom}(V,V) \to \mathbb{C}$ given by the Trace. This is actually not only linear but $\mathfrak{g}$ linear for any liealgebra $\mathfrak{g}$ operating on V provided $\mathbb{C}$ is given the trivial structure. To see this in general we make use of the identity $V^* \times V = \mathrm{Hom}(V,V)$ for which each decomposable element $v^* \otimes v$ is mapped onto the map $x \mapsto v^*(x)v$. The trace of this map is easily computed to be equal to $v^*(v)$.

We have by definition

$$g(v^* \otimes v) = (gv^* \otimes v + v^* \otimes gv)$$

Thus evaluating trace gives

$$\mathrm{Tr}(g(v^* \otimes v)) = (gv^*(v) + v^*(gv) = v^*(-gv) + v^*(gv) = 0$$

The kernel of the trace map is hence a submodule which hence should be identified with $S^2(V)$ the traceless matrices.

We also see that $\bigwedge^2(V)$ is trivial, thus the alternating bilinear forms (unique up to a scalar) on V are $\mathfrak{sl}(2)$ invariant, and it must be those that effect the isomorphism between V and V^* as we see that $S^2(V)$ is irreducible and hence do not contain any invariant elements.

We can now in general decompose

$$S^n(V) \otimes S^m(V) = \bigoplus_{\alpha=0}^{m} S^{n+m-2\alpha}(V)$$

One of the projections $S^n(V) \otimes S^m(V) \to S^{n+m}(V)$ can be given a natural interpretation as the multiplication of binary forms. The other projections are a little bit more subtle.

Let V be the standard 2-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. To this there will be a corresponding representation of the Lie-group $SL(2,\mathbb{C})$ in the obvious way. Both representations induce actions on $P(V) = \mathbb{C}P^1$ which in the latter case coincides with the standard representation of the group of Moebius transformations $PSL(2,\mathbb{C})$ on the Riemannsphere $\mathbb{C}P^1$.

To each Moebius transformation A is associated two fixed points, those are given by the eigenvectors to the corresponding action on V. In the same way to each element $a \in \mathfrak{sl}(2)$ there are two corresponding eigenvectors, which corresponds to the fixed points on the Riemann sphere.

Conversely given any two fixed points, there is a 1- parameter subgroups fixing them. It is given by the exponentiation of the line of elements $ta \in \mathfrak{sl}(2)$ with the given eigenspaces.

There are two types of elements. Either the fixedpoints are distinct, or they coincide. The first type corresponds to semi- simple elements the second to nilpotent elements.

In the action of $\mathfrak{sl}(2)$ on V all elements v are equivalent. Each v determines either a 2-dimensional subgroup of $PSL(2,\mathbb{C})$ or a 2-dimensional subalgebra of $\mathfrak{sl}(2)$ given by the projective stabilisator $Sv = \{a : av = \lambda v\}$. (The stabilisator of the action on $P(V)$) It contains a distinguished element (up to a multiplicative scalar) - the uni-(nil)potent element (corresponding to $\lambda = 1, (0)$). Any two Sv_1 and Sv_2 have a non-empty intersection, the 1-parameter subgroup generated by the fixed points v_1, v_2.

The un-ordered pairs of points on $\mathbb{C}P^1 = P(V)$ are parametrised by a $\mathbb{C}P^2$, more precisely by $P(S^2(V))$, the linear system of binary quadrics, via its zeroes. This $\mathbb{C}P^2$ comes equipped with a canonically given conic, corresponding to the nilpotent elements.

This projective plane can also be thought of as parametrising all 1-parameter subgroups of $PSL(2,\mathbb{C})$ or equivalently, all vector fields, up to multiplication with a scalar. Thus the lifting to $\mathfrak{sl}(2)$ gives a parametrisation of all complex vector fields on the Riemann-sphere.

Clearly the lie-algebra structure on $S^2(V)$ is the one given by the identification of this space with $\mathfrak{sl}(2)$. It can be useful to explain this in detail.

Recall that we have a map $\pi : V \otimes V \to \bigwedge^2(V) \equiv \mathbb{C}$ given by $x \otimes y \mapsto x \wedge y$. The kernel of this map is obviously given by $S^2(V)$. Now recall that the wedge product on V determines a $\mathfrak{sl}(2)$ invariant bilinear form on V which effects the isomorphism with V with its dual V^*. Hence we can write down an identification of $V \otimes V$ with $\mathrm{Hom}(V,V)$ via $v \otimes w \mapsto (x \mapsto (v \wedge x)w)$. The trace of this mapping is clearly $v \wedge w$ which actually identifies π with the Trace.

A basis for $S^2(V)$ is given by $x^2 = x \otimes x, xy = \frac{1}{2}(x \otimes y + y \otimes x), y^2 = y \otimes y$ given a basis x, y for V. It is natural to chose the latter basis as self- dual (i.e. letting $x \wedge y = 1$). Via the identification of $V \times V$ with $\mathrm{Hom}(V,V)$ we will get matrix representations for the elements x^2, xy, y^2 as follows

$$x \otimes x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$x \otimes y \mapsto \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$y \otimes y \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Thus to each binary form $ax^2 + bxy + cy^2$ we associate the element $\begin{pmatrix} -\frac{b}{2} & a \\ -c & \frac{b}{2} \end{pmatrix}$.

Notice that the discriminant $4ac - b^2$ corresponds to the determinant (up to a trivial factor 4) of the matrix.

By this transportation of structure we get the liealgebra structure on $S^2(V)$. Set $X = x^2, Y = y^2$ and $Z = xy$ and we get $[Z, X] = -X, [Z, Y] = Y, [X, Y] = 2Z$.

The adjoint action of $\mathfrak{sl}(2)$ on $S^2(V)$ has more structure. As we noted there is a distinguished conic D (the discriminant conic of nilpotent elements). Given any semi-simple element H its polar with respect to D will intersect the conic in two points X, Y corresponding to the eigenvectors of the adjoint map $ad(H)$. Conversely given any line L in $P(S^2(V)$ it is in general not a subalgebra, in fact $[L, L]$ will correspond to the polar of the line. Thus the subalgebras will correspond exactly to the lines tangent to the distinguished conic D. We hence get a geometric picture of all the subalgebras of $\mathfrak{sl}(2)$. The 1-dimensional ones will correspond to the points, and the 2-dimensional to the tangents to D.

As a vectorspace $S^2(V)$ is clearly not homogenous with respect to the action. Given an element $v \in \mathfrak{sl}(2)$ its stabilisator Sv will depend on what type of element v is. If v is semi-simple (i.e $v \notin D$) then S_v is simply the subalgebra spanned by

v (i.e. the point v itself considered projectively). But if $v \in D$ then Sv coincides with the tangent line.

As the reader already must have suspected the conic D is $\mathfrak{sl}$ invariant. (In fact it is given by the Killing form on $\mathfrak{sl}(2)$)$S^2(V)$ is irreducible, but the space $S^2(S^2(V)$ is not. In fact given

$$S^2V \otimes S^2(V) = S^4(V) \oplus S^2(V) \oplus S^0(V)$$

we can identify

$$S^2(S^2(V) \otimes S^2(V)) = S^4(V) \oplus S^0(V)$$
$$\bigwedge^2(S^2(V)) = S^2(V)$$

Clearly $S^0(V) = \mathbb{C}$ is identified with the space spanned by the invariant conic D. Note that the isomorphism between $S^2(V)$ and its dual $S^2(V^*)$ is now effected by D which is symmetric.

Thus the projection $S^2(V) \otimes S^2(V) \to \mathbb{C}$ is given by evaluation of the quadratic form D. Conversely given any other quadratic form Q complementary to D we can evaluate Q by plugging in x^2, y^2 and xy appropriately and get a binary quartic, an element of $S^4(V)$. Finally the projection map $S^2(V) \otimes S^2(V) \to \bigwedge^2(S^2(V)) \equiv S^2(V)$ recaptures the adjoint representation.

Exercises

1 Given a $\mathfrak{sl}(2)$ representation W with weightfunction w. Show that the dimension of the trivial subrepresentation is given by

$$w(0)^2 - w(1)^2 + 2\sum_{\alpha>0} w(\alpha)^2 - w(1+\alpha)w(1-\alpha)$$

and check that this expression is always non-negative. Give conditions when it is zero.

2 Let V be the standard representation of $\mathfrak{sl}(2)$. Decompose the following modules into irreducible components
i) $S^2(V) \otimes S^2(V) \otimes S^2(V)$
ii) $S^3(S^2(V))$
iii) $S^2(S^3(V))$
iv) $S^3(S^3(V))$

3 With V as in the previous exercise. Given the decomposition

$$S^2(V) \otimes V = S^3(V) \oplus V$$

Show that the projection $S^2(V) \otimes V \to V$ gives the standard representation of $\mathfrak{sl}(2)$ on V.

In the same vein given

$$S^2(V) \otimes S^2(V) = S^4(V) \oplus S^2(V) \oplus \mathbb{C}$$

Interpret the projections
$S^2(V) \otimes S^2(V) \to S^2(V)$ and $S^2(V) \otimes S^2(V) \to \mathbb{C}$