Advanced linear and multilinear algebra (MMA200)

Time: 2016-01-07, 14:00-18:00.

Tools: No calculator or handbook is allowed.

Questions: Jakob Hultgren, 0703-088304

Grades: Each problem gives 6 points. At most 6 bonus points from the exercise sessions will be added to the result. Grades are G (15-24 points) and VG (25-30).

---

1 Find all abelian groups of order 162. Give both the primary decomposition and the invariant factor decomposition of each group.

2 Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the Jordan canonical form and the minimal polynomial of A.

3 Let $v_1, \dots, v_m$ be linearly independent elements of a finite-dimensional vector space and let $u \in \bigwedge^k V$ be such that $u \wedge v_j = 0$ for each $j = 1, \dots, m$. Prove that $k \geq m$ and that $u = v_1 \wedge \cdots \wedge v_m \wedge w$ for some $w \in \bigwedge^{k-m} V$.

4 Prove that if V is a complex representation of a finite group G, and W is a subrepresentation of V, then there exists a subrepresentation W' with $V = W \oplus W'$.

5 A finite group G has six conjugacy classes, which we denote $C_1, \dots, C_6$. We know two irreducible characters χ_1 and χ_2 of the group, given by

	C_1	C_2	C_3	C_4	C_5	C_6
χ_1	1	-1	1	-1	i	$-$i
χ_2	2	2	-1	-1	0	0

Compute the whole character table of G. Also determine the number of commutators in G, that is, the number of elements of the form $aba^{-1}b^{-1}$, with a, $b \in G$.

Advanced linear and multilinear algebra (MMA200)

2016-01-07, Solutions

1 Find all abelian groups of order 162. Give both the primary decomposition and the invariant factor decomposition of each group.

Since $162 = 2 \cdot 3^4$, the groups are enumerated by the partitions of 4. There are 5 such partitions, namely,

$$4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1.$$

These correspond to the groups

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_{81} &\simeq \mathbb{Z}_{162}, \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{27} &\simeq \mathbb{Z}_3 \times \mathbb{Z}_{54}, \\ \mathbb{Z}_2 \times \mathbb{Z}_9^2 &\simeq \mathbb{Z}_9 \times \mathbb{Z}_{18}, \\ \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_9 &\simeq \mathbb{Z}_3^2 \times \mathbb{Z}_{18}, \\ \mathbb{Z}_2 \times \mathbb{Z}_3^4 &\simeq \mathbb{Z}_3^3 \times \mathbb{Z}_6, \end{aligned}$$

respectively, where we have given first the primary decomposition and then the invariant factor decomposition.

2 Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the Jordan canonical form and the minimal polynomial of A.

It is easy to compute $\det(A - \lambda I) = \lambda^2(\lambda - 1)^2$, so the eigenvalues are 0 and 1. Both eigenvalues are double, so $\lambda \in \{0, 1\}$ corresponds either to a 2×2 Jordan block $\left[\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right]$ or to the sum of two 1×1-blocks, $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$. We can distinguish these cases by noting that the dimension of the corresponding eigenspace is 1 and 2, respectively. Starting with the case $\lambda = 0$, the eigenvalue equation $Ax = 0x$ has the solution space $x_1 = x_2 = x_4 = 0$, which is a line (the x_3-axis). For $\lambda = 1$, the equation $Ax = 1x$ is equivalent to

$$\begin{cases} -2x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 - x_4 = 0. \end{cases}$$

This is a plane (we can choose x_3 and x_4 arbitrarily and solve for x_1, x_2). We conclude that A has Jordan canonical form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(the answer is not unique since we may permute the blocks). It follows that the minimal polynomial is $p(\lambda) = \lambda^2(\lambda - 1)$.

3 Let $v_1, \ldots, v_m$ be linearly independent elements of a finite-dimensional vector space and let $u \in \bigwedge^k V$ be such that $u \wedge v_j = 0$ for each $j = 1, \ldots, m$. Prove that $k \geq m$ and that $u = v_1 \wedge \cdots \wedge v_m \wedge w$ for some $w \in \bigwedge^{k-m} V$.

Let $n = \dim V$. We may complete $v_1, \ldots, v_m$ to a basis $v_1, \ldots, v_n$ for V. Then, $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$ is a basis for $\bigwedge^k V$, where $I = \{i_1, \ldots, i_k\}$ runs over subsets of $\{1, \ldots, n\}$ of size k, with $i_1 < \cdots < i_k$. Expanding u in this basis, we can write

$$u = \sum_I u_I \, v_I.$$

The condition $u \wedge v_j = 0$ gives

$$0 = \sum_{j \notin I} u_I \, v_I \wedge v_j.$$

Since the terms are linearly independent, it follows that $u_I = 0$ for $j \notin I$. This holds for $j \in [m] = \{1, \ldots, m\}$, so we can write

$$u = \sum_{[m] \subseteq I} u_I \, v_I.$$

Since the sets I have size k, it follows that indeed $k \geq m$ and $u = v_1 \wedge \cdots \wedge v_m \wedge w$ for some w.

4 Prove that if V is a complex representation of a finite group G, and W is a subrepresentation of V, then there exists a subrepresentation W' with $V = W \oplus W'$.

See the course literature. For instance, two distinct proofs are given in the lecture notes of Ulf Persson.

5 A finite group G has six conjugacy classes, which we denote $C_1, \ldots, C_6$. We know two irreducible characters χ_1 and χ_2 of the group, given by

	C_1	C_2	C_3	C_4	C_5	C_6
χ_1	1	-1	1	-1	i	$-$i
χ_2	2	2	-1	-1	0	0

Compute the whole character table of G. Also determine the number of commutators in G, that is, the number of elements of the form $aba^{-1}b^{-1}$, with $a, b \in G$.

Recall that any group has a trivial conjugacy class C_{triv} containing only the identity. Moreover, if χ is the character of a d-dimensional representation, then $\chi(C_{\text{triv}}) = d$. It follows that $C_{\text{triv}} = C_1$, since all other columns contain non-positive entries. Moreover, χ_1 is a 1-dimensional character. As we have discussed during the course, if χ is any irreducible character and ψ is a one-dimensional character, then $\chi\psi$ is again an irreducible character (it is the character of the tensor product, which is irreducible by a standard criterion). In the case at hand, we can use this to find the characters χ_1^2, χ_1^3, χ_1^4 (which is the trivial character) and $\chi_1\chi_2$. Since we now have as many irreducible characters as conjugacy classes, we have found the complete character table

	C_1	C_2	C_3	C_4	C_5	C_6
χ_1^4	1	1	1	1	1	1
χ_1^2	1	1	1	1	-1	-1
χ_1	1	-1	1	-1	i	$-$i
χ_1^3	1	-1	1	-1	$-$i	i
χ_2	2	2	-1	-1	0	0
$\chi_1\chi_2$	2	-2	-1	1	0	0 .

For the last question, we use the fact that the number of one-dimensional representations is $|G|/|G'|$. Moreover, $|G|$ is the sum of the squares of the dimensions of all irreducible representations. In our case, $|G| = 4 \cdot 1^2 + 2 \cdot 2^2 = 12$ and we have 4 one-dimensional representations. Thus, $|G'| = 3$ so $G' = \{1, x, x^{-1}\}$ for some $x \in G$. Since G' is generated by all commutators, either x or x^{-1} is a commutator. But if $x = aba^{-1}b^{-1}$ is a commutator, so is $x^{-1} = bab^{-1}a^{-1}$. We conclude that there are exactly three commutators.