Additional notes on group representations
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Throughout, group means finite group and representation means finite-dimensional representation over $\mathbb{C}$.

Interpreting the character table

Let $\mathrm{Irr}(G)$ denote the non-equivalent irreducible characters of G and $\mathrm{Cl}(G)$ the conjugacy classes. Then, the square matrix $(\chi(C))_{\chi\in\mathrm{Irr}(G),\, C\in\mathrm{Cl}(G)}$ is called the character table of G. As we have seens, rows and colums satisfy the orthogonality relations

$$\delta_{\chi,\psi} = \sum_{C\in\mathrm{Cl}(G)} \frac{|C|}{|G|}\,\chi(C)\overline{\psi(C)}, \tag{1a}$$

$$\frac{|G|}{|C|}\,\delta_{C,D} = \sum_{\chi\in\mathrm{Irr}(G)} \chi(C)\overline{\chi(D)}. \tag{1b}$$

Although a group is not uniquely determined by its character table, a lot of information about it is. In particular, we can read off all normal subgroups and identify the center and the commutator subgroup.

To see how this is done, let N be a normal sungroup of G and consider a representation of G/N, that is, a homomorphism from G/N to $\mathrm{GL}(V)$. By generalities on normal subgroups, there is a bijection between representations of G/N and representations π of G such that $N\subseteq \mathrm{Ker}(\pi)$. By Lemma 1 below, $\mathrm{Ker}(\pi)$ is easily obtained from the character $\chi=\mathrm{Tr}(\pi)$. For this reason, we will write $\mathrm{Ker}(\chi)$ instead of $\mathrm{Ker}(\pi)$. It is easy to see that a representation of G/N is irreducible if and only if the corresponding representation of G is. Thus,

$$\mathrm{Irr}(G/N) = \{\chi\in\mathrm{Irr}(G);\ N\subseteq \mathrm{Ker}(\chi)\}. \tag{2}$$

Lemma 1. *For any representation π with character χ,*

$$\mathrm{Ker}(\pi) = \{g\in G;\ \chi(g)=\chi(1)\}.$$

Proof. If $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $\pi(g)$ we need to prove that

$$\lambda_1 + \cdots + \lambda_m = m \quad \Longleftrightarrow \quad \lambda_1 = \cdots = \lambda_m = 1.$$

As we have observed before, the eigenvalues λ_j are roots of unity. In particular, $\mathrm{Re}(\lambda_j) \leq 1$ with equality only for $\lambda_j = 1$, which proves the equivalence above. $\square$

If $g \in \mathrm{Ker}(\chi)$ for all χ, then choosing D as the conjugacy class of g and $C = \{1\}$ in (1b) we find that $g = 1$. Applying this fact to the group G/N gives

$$N = \bigcap_{\chi \in \mathrm{Irr}(G),\, N \subseteq \mathrm{Ker}(\chi)} \mathrm{Ker}(\chi). \tag{3}$$

Thus, any normal subgroup is an intersection of kernels of irreducible characters. Since the converse is clear, all normal subgroups can be found from the character table.

Proposition 2. *The normal subgroups of G are precisely the intersections of kernels of irreducible characters.*

Let us now consider the commutator subgroup (or derived group) G'. By definition, it is the smallest normal subgroup containing all elements of the form $aba^{-1}b^{-1}$. Since G is abelian if and only if G' is trivial, $|G|/|G'|$ gives a measure of how commutative G is. More generally, a quotient G/N is abelian if and only if $G' \subseteq N$.

Proposition 3. *The subgroup G' is the intersection of the kernels of all one-dimensional representations of G. Moreover, the number of one-dimensional representations of G is $|G|/|G'|$.*

Proof. By (2),

$$\mathrm{Irr}(G/G') = \{\chi \in \mathrm{Irr}(G);\ G' \subseteq \mathrm{Ker}(\chi)\}$$

and by (3),

$$G' = \bigcap_{\chi \in \mathrm{Irr}(G),\, G' \subseteq \mathrm{Ker}(\chi)} \mathrm{Ker}(\chi).$$

Thus, it suffices to show that a representation of G is one-dimensional if and only if it can be projected to G/G'. Since G/G' is abelian, any representation of G/G' is one-dimensional. Conversely, if π one-dimensional, then $\pi(aba^{-1}b^{-1}) = \pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1} = 1$, so π projects to G/G'. $\square$

Next we consider the center. Recall that $Z(G)$ is the set of elements in G that commute with all other elements; it is an abelian normal subgroup. Since G is abelian if and only if $Z(G) = G$, $|G|/|Z(G)|$ gives a measure of how non-commutative G is.

When χ is a character, we define

$$Z(\chi) = \{g \in G;\ |\chi(g)| = \chi(1)\}.$$

Again using that the eigenvalues of $\pi(g)$ are roots of unity, this can only happen when they are all equal, so we can also write

$$Z(\chi) = \{g \in G;\ \pi(g) \in \mathbb{C}\,\mathrm{Id}\}, \qquad \chi = \mathrm{Tr}(\pi).$$

The reason for the notation is the following fact.

Lemma 4. *The center of G is $Z(G) = \bigcap_{\chi \in \mathrm{Irr}(G)} Z(\chi)$.*

Proof. Let π be an irreducible representation on the vector space V and χ its character. If $g \in Z(G)$ then $\pi(g)\pi(h) = \pi(h)\pi(g)$ for all $h \in G$. Thus, $\pi(g) \in \mathrm{End}_G(V)$. By Schur's lemma, $g \in Z(\chi)$. Conversely, if $\pi(g) = \lambda\,\mathrm{Id}$, then for any $h \in G$ we have $\pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = \mathrm{Id}$, which implies $ghg^{-1}h^{-1} \in \mathrm{Ker}(\pi)$. If this holds for each irreducible representation π, then (3) with $N = \{1\}$ gives $ghg^{-1}h^{-1} = 1$, so $g \in Z(G)$. $\square$

Example. Suppose G has the character table

$$\begin{array}{ccccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & -1 & 1 & -1 & \mathrm{i} & -\mathrm{i} \\
\chi_3 & 1 & 1 & 1 & 1 & -1 & -1 \\
\chi_4 & 1 & -1 & 1 & -1 & -\mathrm{i} & \mathrm{i} \\
\chi_5 & 2 & 2 & -1 & -1 & 0 & 0 \\
\chi_6 & 2 & -2 & -1 & 1 & 0 & 0
\end{array}.$$

We follow the convention that first column corresponds to the conjugacy class $C_1 = \{1\}$ and the first row to the trivial representation. This can be seen immediately since, by (1), they are the unique row and column with only positive entries. The entries of the first column are then the dimensions of the corresponding representations. In this case, $\chi_1, \ldots, \chi_4$ are one-dimensional and χ_5, χ_6 two-dimensional. The order of the group is thus $4 \cdot 1^2 + 2 \cdot 2^2 = 12$.

We can then compute the order of the conjugacy classes using (1b) and find that $|C_1| = |C_2| = 1$, $|C_3| = |C_4| = 2$, $|C_5| = |C_6| = 3$. We may check again that the total number of elements is $2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 = 12$. Writing for short $C_{i_1 \cdots i_k} = C_{i_1} \cup \cdots \cup C_{i_k}$, the kernels of the characters are $\operatorname{Ker}(\chi_1) = C_{123456} = G$, $\operatorname{Ker}(\chi_2) = \operatorname{Ker}(\chi_4) = C_{13}$, $\operatorname{Ker}(\chi_3) = C_{1234}$, $\operatorname{Ker}(\chi_5) = C_{12}$, $\operatorname{Ker}(\chi_6) = C_1$. Taking intersections of these does not lead to any further subgroups; for instance, $C_{13} \cap C_{12} = C_1$. Thus, G has exactly five normal subgroups. Moreover $G' = \bigcap_{j=1}^{4} \operatorname{Ker}(\chi_j) = C_{13}$. To compute the center, we see at a glance that $Z(\chi) = G$ for the one-dimensional representations and $Z(\chi) = C_{12}$ for the two-dimensional ones. Thus, $Z(G) = C_{12}$. In conclusion, the lattice of normal subgroups looks like

$$C_1 = \{1\} \subseteq \begin{Bmatrix} C_{12} = Z(G) \\ C_{13} = G' \end{Bmatrix} \subseteq C_{1234} \subseteq C_{123456} = G.$$

We remark that, in this example, the character table does not determine the group. There are five groups of order 12, two of which have the table above as their character table (one of them is the symmetries of a regular hexagon).

The group algebra

Recall that the group algebra $\mathbb{C}[G]$ of G is the complex vector space with G as a basis. Equivalently, it is the space of all functions from G to $\mathbb{C}$. The correspondence between these two descriptions is that a linear combination $\sum_{g \in G} a_g g$ corresponds to the function $f(g) = a_g$; in particular, the generator g to the delta function $\delta_g(h) = \delta_{g,h}$. We multiply two elements of the group algebra simply by using the multiplication in G:

$$\sum_g a_g g \sum_g b_h h = \sum_{g,h} a_g b_h gh.$$

In the description as functions, this is a convolution:

$$(\phi\psi)(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g).$$

Left multiplication $\pi(g)h = gh$ extends to a representation of G on $\mathbb{C}[G]$. This is the *regular representation.* We have already seen that

$$\mathbb{C}[G] \simeq \bigoplus_{V \in \operatorname{Irr}(G)} (\dim V)\, V \tag{4}$$

(here we use $\mathrm{Irr}(G)$ to denote the representation spaces rather than their characters). This does not say anything about the *algebra* structure on $\mathbb{C}[G]$. However, we can easily obtain a refinement of (4) that does. To this end, we consider $\mathbb{C}[G]$ as a representation of $G \times G$ ny the rule $\pi(g,h)k = ghk^{-1}$. Note that if V is any representation of G, $\mathrm{End}(V)$ is a representation of $G \times G$ under $\pi(g,h)\phi = \pi(g) \circ \phi \circ \pi(h^{-1})$. Moreover, $\mathrm{End}(V)$ is an algebra (under composition of maps). This gives a meaning to the following result.

Theorem 5. *As a representation of $G \times G$ and as an associative algebra,*

$$\mathbb{C}[G] \simeq \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V).$$

Proof. A representation of V is a homomorphism π from G to $\mathrm{GL}(V)$. It extends by linearity to a map $\pi_V : \mathbb{C}[G] \to \mathrm{End}(V)$. It is trivial to check that π_V commutes with the action of $G \times G$ and preserves multiplication. Thus, it is enough to check that $\Phi = \bigoplus_V \pi_V$ is bijective. If $\Phi(x) = 0$ for some element $x = \sum_g a_g g$, then x acts trivially in any irreducible representation and hence in any representation. But then, in the regular representation, $x = \pi(x)e = 0$. This shows that Φ is injective. The surjectivity follows by comparing dimensions; indeed, by (4), $\dim(\mathbb{C}[G]) = \sum_V \dim(V)^2 = \sum_V \dim \mathrm{End}(V)$. $\square$

We write $L^2(G)$ for $\mathbb{C}[G]$, viewed as a space of functions, with scalar product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

We have previously seen that the characters of irreducible representations form an orthonormal basis for a subspace of $L^2(G)$. We will now prove that the *matrix elements* of irreducible representations form an orthonormal basis for the whole space. By a matrix element, we simply mean an element $\pi(g)_{kl}$ of the matrix for $\pi(g) \in \mathrm{End}(V)$, with respect to some basis of the representation V. To obtain genuine orthogonality relations, we must choose this basis in a good way. Recall that any representation carries an invariant Hermitian form (see e.g. p. 3 of Persson's lecture notes; for *irreducible* representations, this form is unique up to multiplication by a positive scalar). Invariant means that $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$. We fix a basis for each irreducible representation which is *orthogonal* with respect to such a form. In terms of the matrix elements, it is easy to see that invariance means that

$$\pi(g^{-1})_{kl} = \overline{\pi(g)_{lk}}. \tag{5}$$

Theorem 6 (Peter–Weyl theorem for finite groups)**.** *With the matrix elements $\pi^V(g)_{ij}$ defined as explained above, the collection of all such elements, for all irreducible representations V, form an orthogonal basis for $L^2(G)$. Explicitly, the orthogonality relation is*

$$\frac{1}{|G|}\sum_{g\in G}\pi^V(g)_{ij}\,\overline{\pi^W(g)_{kl}} = \frac{\delta_{ik}\delta_{jl}\delta_{VW}}{\dim V}. \tag{6}$$

Proof. Fixing the indices (j,k), let $A\in \mathrm{Hom}(V,W)$ be defined by $Ae^V_j = e^W_k$ and $Ae^V_m = 0$ for $m\neq j$. In terms of matrix elements, $A_{xy} = \delta_{xj}\delta_{yk}$. Let $\bar{A}\in \mathrm{Hom}_G(V,W)$ be the average of this operator over the group action, that is,

$$\bar{A} = \frac{1}{|G|}\sum_{g\in G}\pi^W(g^{-1})\circ A\circ \pi^V(g). \tag{7}$$

Then, by Schur's lemma, $\bar{A} = \delta_{V,W}\lambda\,\mathrm{Id_V}$ for some $\lambda\in\mathbb{C}$. In the case $V=W$ we have $\mathrm{Trace}(\bar{A}) = \mathrm{Trace}(A) = \delta_{jk}$, which gives $\lambda = \delta_{jk}/\dim(V)$. We conclude that

$$\bar{A} = \frac{\delta_{VW}\delta_{jk}}{\dim V}\ \mathrm{Id}_V\,. \tag{8}$$

In terms of matrix elements, (7) takes the form

$$\bar{A}_{mn} = \frac{1}{|G|}\sum_{g\in G}\sum_{xy}\pi^W(g^{-1})_{mx}A_{xy}\pi^V(g)_{yn} = \frac{1}{|G|}\sum_{g\in G}\pi^W(g^{-1})_{mj}\pi^V(g)_{kn}.$$

By (8), the left-hand side is $\delta_{VW}\delta_{jk}\delta_{mn}/\dim V$. Using (5) on the right-hand side we obtain (6). It remains to prove that the matrix elements span $L^2(G)$. That follows from a dimension count; the total number of matrix elements is $\sum_V \dim(V)^2 = \dim(L^2(G))$. □

Versions of Theorem 6 can be obtained for various classes of infinite groups and is then a basis for harmonic analysis on groups. In particular, the cases $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$ relate to the classical theory of Fourier series and Fourier transform.