Determinants
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This is an incomplete draft, which may contain typos.

Introduction, Hodge star

Exterior algebra is intimately related to the theory of determinants and can be used to prove all their main properties. Rather than rederiving well-known facts such as $\det(AB) = \det(A)\det(B)$, we will use exterior algebra to prove two useful results on determinants that are typically not covered in linear algebra courses. The first is Dodgson's condensation method, the second is the Plücker relations.

Throughout, V will be an n-dimensional vector space over a field K. We write $V^* = \mathrm{Hom}(V, K)$ for the dual space. As is customary, we write $\langle x, y\rangle$ instead of $y(x)$ when $x \in V$ and $y \in V^*$.

When $A \in \mathrm{Hom}(V, W)$ we write $A^* \in \mathrm{Hom}(W^*, V^*)$ for the dual map defined by

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

Assume that we have chosen a basis e for the one-dimensional space $\bigwedge^n V$. Then, if $x \in \bigwedge^m V$ and $y \in \bigwedge^{n-m} V$, we have $y \wedge x = ce$ for some $c \in K$. For fixed x, the constant c depends linearly on y. Thus, we can define a map $\phi : \bigwedge^m V \to \bigwedge^{n-m} V^*$ by

$$y \wedge x = \langle y, \phi(x)\rangle e, \qquad x \in \bigwedge^m V, \quad y \in \bigwedge^{n-m} V.$$

(Here we are using the canonical isomorphism $(\bigwedge^k V)^* \simeq \bigwedge^k (V^*)$.) The map ϕ is often called the Hodge star and written $*x$ rather than $\phi(x)$.

Let $(e_j)_{j=1}^n$ be a basis for V and $(e_j^*)_{j=1}^n$ the dual basis. We may take $e = e_1 \wedge \cdots \wedge e_n$. Then, it is easy to check that

$$\phi(e_{k_1} \wedge \cdots \wedge e_{k_m}) = \varepsilon\, e_{l_1}^* \wedge \cdots \wedge e_{l_{n-m}}^*, \tag{1}$$

where $(k_1, \dots, k_m, l_1, \dots, l_{n-m})$ is a permutation of $(1, \dots, n)$ and $\varepsilon \in \{\pm 1\}$ is the signature of that permutation. In particular, ϕ maps a basis to a basis, so it is an isomorphism of vector spaces. We stress that it is "almost canonical" in the sense that it does not depend on a choice of basis for V, but only on a choice of basis for the one-dimensional space $\bigwedge^n V$.

Geometrically, the choice of the element e means that V is equipped with an orientation and a rule for computing n-dimensional volumes (but no rules for computing k-dimensional volumes for $k < n$). Namely, if $v_1, \ldots, v_n$ are any vectors and $v_1 \wedge \cdots \wedge v_n = ae$, then we can define the volume of the parallelepiped spanned by the v_j to be $|a|$, and call them positively oriented if $a > 0$.

Dodgson condensation

We start with the following rather elementary fact.

Proposition 1. *For $A \in \mathrm{End}(V)$ and $0 \leq m \leq n$,*

$$\det(A)\phi = \bigwedge\nolimits^{n-m}(A^*) \circ \phi \circ \bigwedge\nolimits^{m}(A). \tag{2}$$

Proof. By linearity, it's enough to compute

$$\langle v_1 \wedge \cdots \wedge v_{n-m}, X v_{n-m+1} \wedge \cdots \wedge v_n \rangle e,$$

where X are the two sides of (2) and $v_j \in V$. The left-hand side becomes

$$\det(A) v_1 \wedge \cdots \wedge v_n$$

and the right-hand side

$$\begin{aligned}
&\langle v_1 \wedge \cdots \wedge v_{n-m}, \bigwedge\nolimits^{m}(A^*) \circ \phi \circ \bigwedge\nolimits^{n-m}(A) v_{n-m+1} \wedge \cdots \wedge v_n \rangle e \\
&\qquad = \langle A v_1 \wedge \cdots \wedge A v_{n-m}, \phi(A v_{n-m+1} \wedge \cdots \wedge A v_n) \rangle e \\
&\qquad\qquad = A v_1 \wedge \cdots \wedge A v_n = \bigwedge\nolimits^{n}(A) v_1 \wedge \cdots \wedge v_n.
\end{aligned}$$

Since $\bigwedge^n(A)$ is multiplication by $\det(A)$, this completes the proof. □

If $\det(A) \neq 0$, this means that $\bigwedge^m(A)$ is invertible with inverse

$$\frac{1}{\det(A)} \phi^{-1} \circ \bigwedge\nolimits^{n-m}(A^*) \circ \phi. \tag{3}$$

In particular,

$$A^{-1} = \frac{1}{\det(A)} \phi^{-1} \circ \bigwedge\nolimits^{n-1}(A^*) \circ \phi. \tag{4}$$

Working this out in coordinates gives a well-known identity for the matrix inverse. However, we want to use it to find something less familiar. To this end, we note that if A is invertible, then $\bigwedge^m(A^{-1})$ is an inverse of $\bigwedge^m(A)$. Thus, the m-th exterior product of (4) must agree with (3). After replacing A^* by A and m by $n-m$ we arrive at the following result.

Corollary 2. *In the notation above,*

$$\det(A)^{n-m-1} \circ \bigwedge^m(A) = \phi \circ \bigwedge^{n-m}(\phi^{-1} \circ \bigwedge^{n-1}(A) \circ \phi) \circ \phi^{-1}. \tag{5}$$

To be precise, we have only proved this for invertible matrices. Over the complex numbers, it's clear that it holds for any matrix since both sides are polynomials in the matrix elements of A, and the non-invertible matrices form an open subset in the space of all matrices. One can use this to prove that Cor. 2 holds for matrices over any commutative ring, but we will not go into the details.

Let us now write Cor. 2 in coordinate form. As above, pick a basis $(e_j)_{j=1}^n$ such that $e = e_1 \wedge \cdots \wedge e_n$. We write $[n] = \{1, \ldots, n\}$. When $S \subseteq [n]$, we write

$$e_S = e_{s_1} \wedge \cdots \wedge e_{s_m},$$

where s_j are the elements of S written in increasing order. Then, $(e_S)_{S\subseteq[n],\,|S|=m}$ is a basis for $\bigwedge^m V$. Moreover, if

$$Ae_k = \sum_{l=1}^n a_{kl} e_l,$$

then

$$\bigwedge^m(A)(e_S) = \sum_{T\subseteq[n]} \det_{ST}(A) e_T,$$

where we write

$$\det_{ST}(A) = \det_{1\le i,j\le m}(a_{s_i,t_j}),$$

with s_j and t_j the elements of S and T, written in increasing order. That is, the matrix elements of $\bigwedge^m(A)$ are exactly the $m \times m$ *minors* of A. To see this is a straight-forward computation, or a non-computation using defining properties of the determinant. In this notation, (1) takes the form

$$\phi(e_S) = \pm e^*_{S^c}$$

(where the sign depends on S).

Thus, acting by the left-hand side of (5) on a basis element e_S gives

$$\det(A)^{n-m-1} \sum_{|T|=m} \det_{ST}(A) e_T.$$

On the right-hand side, we note that $\phi^{-1}(e_S) = \pm e^*_{S^c}$. Moreover, with $j^c = [n] \setminus j$, we have

$$\phi^{-1} \bigwedge^{n-1}(A) \phi e^*_j = \textstyle\sum_{k=1}^n \pm \det_{j^c,k^c}(A) e^*_k.$$

It follows that the right-hand side gives

$$\pm \sum_{|T|=m} \det_{S^c T^c}(A^\sharp) e_T,$$

where $A^\sharp$ is the matrix with elements $\det_{j^c,k^c}(A)$. Identifying the coefficient of e_T, we obtain the following fact.

Corollary 3 (Jacobi)**.** *For any $n \times n$-matrix A and any $S, T \subseteq [n]$ with $|S| = |T| = m$,*

$$\det(A)^{n-m-1} \det_{ST}(A) = \pm \det_{S^c T^c}(A^\sharp).$$

In future versions of this draft we will be more careful with the sign. Note that, up to multiplying some rows and columns by -1, which only changes the minors by a sign, $A^\sharp$ is the *adjugate matrix*, equal to $\det(A)A^{-1}$ in the invertible case. That's exactly the meaning of (4).

One particularly ineresting case of Corollary 3 is the case $m = n - 2$. Without loss of generality, we may choose $S = T = \{1, \dots, n-2\}$. Writing out the 2×2-determinant on the right-hand side explicitly gives the following result.

Corollary 4 (Dodgson condensation)**.** *The following identity holds:*

$$\det_{1 \le j,k \le n}(a_{jk}) \det_{1 \le j,k \le n-2}(a_{jk}) = \det_{j,k \ne n}(a_{jk}) \det_{j,k \ne n-1}(a_{jk}) - \det_{j \ne n, k \ne n-1}(a_{jk}) \det_{j \ne n-1, k \ne n}(a_{jk}).$$

This is often called the "Lewis Carroll identity", since Dodgson (also known as Carroll) promoted it as a method for computing determinant recursively. As an example, we could compute a 3×3-determinant as

$$1 \times \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}.$$

For numerical determinants, this is much slower than using row and column reductions. However, for determinants with symbolic entries, it is very powerful (see the exercise sheet for two nice examples).

Plücker relations

By a *pure tensor*, we mean an element of the form $u = v_1 \wedge \cdots \wedge v_k$, $v_j \in V$. Pure tensors are geometrically interesting for the following reason. If we declare two non-zero pure tensors to be equivalent if $u = av$ for some $a \in K$, then there is a one-to-one correspondence between the equivalence classes and k-dimensional subspaces in V. Namely, the tensors equivalent to $v_1 \wedge \cdots \wedge v_k$ correspond to $\mathrm{span}\{v_1, \dots, v_k\}$ (Brzezinski, Ex. 5.7). The space of all k-dimensional subspaces of V is called a *Grassmannian*. If we can describe the pure tensors algebraically, we obtain a description of the Grassmannian as an algebraic variety.

It will be convenient to write $\lambda_u(v) = u \wedge v$. If $u \in \bigwedge^k V$, then $\lambda_u : V \to \bigwedge^{k+1} V$. We then have a map $\lambda_{\phi(u)} : V^* \to \bigwedge^{n-k+1} V^*$, whose dual is $\lambda^*_{\phi(u)} : \bigwedge^{n-k+1} V \to V$. Thus, we can introduce the map

$$\mu_u = \lambda_u \circ \lambda^*_{\phi(u)} : \bigwedge^{n-k+1} V \to \bigwedge^{k+1} V.$$

Since μ_u is a quadratic function of u, the following result describes the Grassmannian by quadratic relations, which are known as Plücker relations. As an example, you can check from Theorem 5 that an element

$$u = \sum_{1 \le j < k < 4} a_{jk} e_j \wedge e_k \in \bigwedge^2 (\mathbb{R}^4)$$

is a pure tensor if and only if $u \wedge u = 0$, which amounts to the quadratic identity

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0.$$

Theorem 5. *The element u is a pure tensor if and only if $\mu_u = 0$.*

Proof. Assume first that $u = v_1 \wedge \cdots \wedge v_k$. If the vectors v_j are linearly dependent, then $u = 0$ and hence $\mu_u = 0$. Else, complete the vectors to a basis $(v_j)_{j=1}^n$ for V. Choosing $e = v_1 \wedge \cdots \wedge v_n$, we get from (1) that

$$\phi(u) = v_{k+1}^* \wedge \cdots \wedge v_n^*.$$

We then have, for any $v \in \bigwedge^{n-k+1} V$,

$$\langle \lambda_{\phi(u)}^* v, v_j^* \rangle = \langle v, \phi(u) \wedge v_j^* \rangle = 0, \qquad k+1 \leq j \leq n.$$

It follows that $\lambda_{\phi(u)}^* v \in \operatorname{span}\{v_1, \ldots, v_k\}$, which gives $\mu_u(v) = u \wedge \lambda_{\phi(u)}^* v = 0$.

For the converse, assume that $f_u = 0$ and let $(v_j)_{j=1}^m$ be a basis for $\operatorname{Im}(\lambda_{\phi(u)}^*)$. As before, we complete it to a basis $(v_j)_{j=1}^n$ for V and let $e = v_1 \wedge \cdots \wedge v_n$. That $f_u = 0$ means that $u \wedge v_j = 0$ for $1 \leq j \leq m$. By Lemma 6 below, $u = v_1 \wedge \ldots v_m \wedge w$ for some $w \in \bigwedge^{k-m} V$. By (1), $\phi(u) \in \operatorname{span}\{v_{m+1}^*, \ldots, v_n^*\}$. Suppose first that $\phi(u) \wedge v_j^* \neq 0$ for some $j \geq m+1$. Then, we can pick $x \in \bigwedge^{n-k+1} V$ with $0 \neq \langle x, \phi(u) \wedge v_j^* \rangle = \langle \lambda_{\phi(u)}^* x, v_j^* \rangle$. But this contradicts the fact that $\operatorname{Im}(\lambda_{\phi(u)}^*) = \operatorname{span}\{v_1, \ldots, v_k\}$. It follows that $\phi(u) \wedge v_j^* = 0$ for all $j \geq m+1$. Then, Lemma 6 gives $\phi(u) = v_{m+1}^* \wedge \cdots \wedge v_n^*$ and consequently $u = v_1 \wedge \cdots \wedge v_m$ is a pure tensor (in particular, $m = k$). ☐

In the proof, we used the following result.

Lemma 6. *If $v_1, \ldots, v_m$ are linearly independent elements of V and $u \in \bigwedge^k V$ is such that $u \wedge v_j = 0$ for each j, then $k \geq m$ and $u = v_1 \wedge \cdots \wedge v_m \wedge w$ for some $w \in \bigwedge^{k-m} V$.*

We leave the proof to the reader; one possible starting point is to extend v_j to a basis for the whole space.

If $u = v_1 \wedge \cdots \wedge v_k$ and A is the matrix with columns v_j, then Theorem 5 is a compact way of stating a number of relations for minors of A. I haven't worked out the details, but the result should be

$$\sum_{j \in S^c \cap T} (-1)^j \det_{S \cup j, [n]}(A) \det_{T \setminus j, [n]}(A) = 0.$$

This holds for any $m \times n$-matrix A and any subsets S, $T \subseteq [m]$ with $|S|+1 = |T| - 1 = n$.