Advanced Linear and Multilinear Algebra 2015.
Exercises for 7 September.

1. If H, K and L are subgroups of a group G with $H \subseteq K \cup L$, show that either $H \subseteq K$ or $H \subseteq L$.

2. With $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ the cyclic group with a elements, show that $\mathbb{Z}_a \times \mathbb{Z}_b \simeq \mathbb{Z}_{ab}$ if and only if a and b are relatively prime.

3. By definition, the *center* $Z(G)$ of a group G is the set of elements $a \in G$ such that $ab = ba$ for all $b \in G$.

   (a) Show that the center is a normal subgroup.

   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.

4. (a) Two elements a and b of a group G are called *conjugate* if there exists $g \in G$ such that $a = gbg^{-1}$. Explain in simple terms what it means for two permutations to be conjugate.

   (b) The set of all elements of G that are conjugate to some element $a \in G$ is called a *conjugacy class*. How many conjugacy classes are there in S_5 and how many elements do they have? What can you say about the number of conjugacy classes of S_n?

5. A *Boolean ring* is one where $a^2 = a$ for all a. Show that in a Boolean ring, $a + a = 0$ and $ab = ba$ for all a and b.
   Remark: Any Boolean ring can be realized as (is isomorhic to) a ring where the elements are subsets of some set and the ring operations are

$$a + b = (a \cup b) \setminus (a \cap b), \qquad ab = a \cap b.$$

6. Show that all ideals in $\mathbb{Z}/n\mathbb{Z}$ are principal. For which n is $\mathbb{Z}/n\mathbb{Z}$ a principal ideal domain?

7. Prove that in a principal ideal domain, every non-zero prime ideal is maximal.

8. Let I be the set of polynomials p in $\mathbb{Z}[x]$ such that $p(0)$ is even. Show that I is a maximal ideal. Is it a principal ideal?

9. Prove that $(x^2 - 2)$ is a prime ideal in $\mathbb{Z}[x]$, but not a maximal ideal.

10. Prove that $(x, x^2 + y^2 - 1)$ is not a prime ideal in $\mathbb{R}[x, y]$.