Advanced Linear and Multilinear Algebra 2015.
Exercises for 21 September.

1. Write down all abelian groups of order 360. Give both the primary decomposition and the invariant factor decomposition. Also, compute $\dim_{\mathbb{Z}_2} 2^l G/2^{l+1}G$ for each such G and each non-negative integer l.

2. Consider the positive integers less than and relatively prime to 21. They form a group under multiplication modulo 21. What is the primary decomposition of this group?

3. Give an example of a torsion module over $\mathbb{Z}$ with trivial annihilator (informally: any element is killed by something, but nothing kills all elements).

4. Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Find the primary and rational canonical forms for A, over the fields $\mathbb{Q}$ and $\mathbb{R}$.

5. Write down all possible Jordan canonical forms for matrices with characteristic polynomial $(x-1)^3(x+1)^2$. What is the minimal polynomial in each case?

6. In the lecture notes we gave two methods for constructing the Jordan canonical form of a matrix A from the numbers $c_k = \dim(N(A-\lambda I)^k)$. On the one hand, the number of λ-blocks of size k is $d_{k-1} - d_k$, where $d_k = c_{k+1} - c_k$. On the other hand, c_k counts the k right-most columns in each λ-block. Show that these two methods give the same result.

7. Consider a system of ODEs

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n, \\ &\cdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n, \end{aligned}$$

   where $A = (a_{jk})_{j,k=1}^n$ is a matrix of real constants. Without going into any detailed computation, explain how writing A in Jordan canonical form helps to understand what the solutions look like. For instance, what does the size of the Jordan blocks mean for the solutions? What do complex eigenvalues mean?

8. Suppose the matrix A has minimal polynomial $p(x) = (x-1)^2$. Find an explicit expression for A^k as a linear combination of A and I. Can you use this to find a matrix B with $B^2 = A$?