Advanced Linear and Multilinear Algebra 2015.
Exercises for 5 October.

1. Define $\phi : \mathbb{Z}_2 \to \mathrm{GL}(\mathbb{Z}_2^2)$ by $\phi(0) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\phi(1) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Show that ϕ is a representation over the field $\mathbb{Z}_2$. Show that ϕ is not a sum of irreducible representations. (This shows that Maschke's theorem doesn't hold over any field.)

2. Show that $\chi(g) = \chi(h)$ for all irreducible representations χ if and only if g and h belong to the same conjugacy class.

3. Using the character table of the group S_3 (which we have discussed as the lecture), find the decomposition of $\otimes^n V$, where V is the two-dimensional representation.

4. Find the character table of the symmetric group S_4. (Using our discussion of S_3 you should be able to guess three irreducible representations; the others can be found by taking tensor product of these three and/or exploiting orthogonality relations for characters).

5. If χ is an irreducible character, show that $Z(\chi)/\operatorname{Ker}(\chi)$ is a cyclic group. Deduce that if G is a group which has a faithful irreducible representation (faithful means that $\pi : G \to \mathrm{GL}(V)$ is injective) then $Z(G)$ is cyclic.

6. Find all irreducible representations of the group $G = \mathbb{Z}/n\mathbb{Z}$. Consider the function $x \in L^2(G)$ defined by
$$x(j) = \begin{cases} 1, & 0 \le j \le k-1, \\ 0, & k \le j \le n-1. \end{cases}$$
Find the scalar product $\langle x, \chi \rangle$ for each irreducible character χ (this is called the finite Fourier transform of x). Use the result to compute the sum
$$\sum_{j=1}^{n-1} \left(\frac{\sin(\pi jk/n)}{\sin(\pi j/n)} \right)^2 .$$

7. Let G be a group with character table given on the next page. What is the order of G? Find all normal subgroups of G and identify the commutator subgroup and the center.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	-1	1	1	-1
χ_3	1	1	1	1	1	-1	-1	-1	-1	-1
χ_4	1	1	-1	-1	1	-1	1	-1	-1	1
χ_5	2	-1	0	0	2	2	0	-1	2	0
χ_6	2	-1	0	0	2	-2	0	1	-2	0
χ_7	3	0	-1	1	-1	3	1	0	-1	-1
χ_8	3	0	1	-1	-1	3	-1	0	-1	1
χ_9	3	0	-1	1	-1	-3	-1	0	1	1
χ_{10}	3	0	1	-1	-1	-3	1	0	1	-1