Advanced Linear and Multilinear Algebra 2015.
Exercises for 12 October.

1. A group of people is arriving at a party, each of them carrying an umbrella. On their way home, they all pick up an umbrella at random. How many people are expected to return home carrying their own umbrella? (To be precise, we are asking for the expected value, that is, the arithmetic mean value over all possible outcomes.)

2. Prove that the representation of S_n corresponding to the partition $(n-1,1)$ is equivalent to the standard representation.

3. Let V be an n-dimensional vector space. Then, S_3 acts on $\bigotimes^3 V$ by permuting the factors. Give the decomposition of $\bigotimes^3 V$ into irreducible representations of S_3.

4. Prove the formula
$$\dim(V_\lambda) = \frac{n!}{l_1! \cdots l_m!} \prod_{1 \leq i < j \leq m} (l_i - l_j)$$
given in the lecture notes.

5. Using the above exercise, prove Corollary 4 in the lecture notes. (One way is to consider what happens when a new column is added on the left of the Young diagram.)

6. Consider the formal power series
$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \log \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$
Using known results on partitions, prove the formal power series identity
$$\exp\left(\log \frac{1}{1-x}\right) = \frac{1}{1-x}.$$

7. Let $(12 \dots n) \in S_n$ be a cycle of maximum length. Compute $\chi_\lambda(12 \dots n)$ for all partitions λ.

8. If σ is a product of k disjoint cycles and λ a partition with $\lambda_{k+1} \geq k$, show that $\chi_\lambda(\sigma) = 0$.