Groups of order 6
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The only groups of order 6 are the cyclic group C_6 and the symmetric group S_3. We will show this in an elementary way.

Recall that the order of an element $a \in G$ is the smallest positive integer m such that $a^m = 1$. Then, a generates a subgroup $\langle a \rangle$ of G isomorphic to C_m. By Lagrange's theorem, m divides $|G|$. In our case, we conclude that all elements have order 1, 2, 3 and 6. Clearly, only the trivial element $1 \in G$ has order 1. Note also that if a has order 6, then a^2 has order 3 and a^3 has order 2. Thus, one of the following cases hold:

(a) All non-trivial elements have order 2.

(b) All non-trivial elements have order 3.

(c) There is an element a of order 2 and an element b of order 3.

By Cauchy's theorem, if p is a prime that divides $|G|$, then G has an element of order p. This is often deduced from more general theorems of Sylow. Thus, cases (a) and (b) are impossible. However, we find it instructive to prove this directly by elementary means.

Case (a) means that $a^{-1} = a$ for each $a \in G$. This gives

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$$

so G is abelian. Thus, if a is a non-trivial element, $H = \langle a \rangle$ is a normal subgroup of G. Then, G/H is a group of order 3 and thus isomorphic to C_3. This means that if $b \notin H$, $G/H = \{H, bH, b^2H\}$. Since this contradicts $b^2 = 1$, case (a) is impossible.

In case (b), $a^{-1} = a$ only for $a = 1$, so we can write the elements of G as

$$1,\ a,\ a^{-1},\ b,\ b^{-1}, \dots .$$

In particular, $|G|$ is odd so (b) is impossible.

In case (c), it is easy to check that the elements

$$1,\ b,\ b^2,\ a,\ ab,\ ab^2 \tag{1}$$

are all distinct. Thus, these are all the elements of G. It's easy to check that ba is distinct from the first four elements in (1). Thus, either $ba = ab$ or $ba = ab^2$. Imposing one of these relations, we can take any product involving the elements a and b and move the symbol b to the right, eventually arriving at one of the elements (1). Thus, the group G is determined by the choice of ba. Since we already know two groups of order 6, C_3 and S_3, we conclude that there are no others. It's easy to check that the choice $ba = ab$ corresponds to C_3 (indeed, ab is an element of order 6) and that $ba = ab^2$ corresponds to S_3 (indeed, the elements $a = (12)$ and $b = (123)$ generate S_3 and satisfy $a^2 = b^3 = 1$, $ba = ab^2$).