Representations of the symmetric group
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We will describe all irreducible representations of the group S_n and compute the character table. There are several approaches for doing this. Our main source has been the excellent book "Representation Theory" by Fulton and Harris, which we can recommend for further reading and an unusually large number of instructive examples. We will state and explain all main results before going into the proofs. As before, when we make general statements about group representations, we mean finite-dimensional representations over $\mathbb{C}$ of finite groups. The group algebra $\mathbb{C}[S_n]$ will be denoted $\mathbf{A}_n$.

Statement of results

Let us start with some remarks relevant for general groups. We have seen that

$$\mathbb{C}[G] \simeq \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V), \tag{1}$$

where the isomorphism is both in the sense of algebras and in the sense of representations of G. Choosing a basis for V, we can view $\mathrm{End}(V)$ as a ring of matrices. The group action is then given by matrix multiplication $\pi_V(g)A$, or equivalently

$$\pi_V(g)[A_1 \ \cdots \ A_n] = [\pi_V(g)A_1 \ \cdots \ \pi_V(g)A_n],$$

where A_j are the columns of A. Hence, the representation V can be identified with the space of matrices that have non-zero entries only in (say) the first column. This space equals $\mathrm{End}(V)E_{11}$, where $E_{11} \in \mathrm{End}(V)$ is the matrix with 1 in the upper left corner and 0 elsewhere. Note that $E_{11}^2 = E_{11}$, that is, E_{11} is an idempotent. If Φ is an isomorphism from the right-hand side of (1) to the left, it follows that $\Phi(E_{11})$ is an idempotent in $\mathbb{C}[G]$ and $V \simeq \mathbb{C}[G]\Phi(E_{11})$. This proves the following fact.

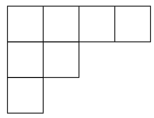
Proposition 1. *Any irreducible representation V of G is equivalent to a left ideal $\mathbb{C}[G]e$, where e is an idempotent.*

As a simple example, if $c = \sum_{g\in G} g$ then $hc = \sum_{g\in G} hg = \sum_{g\in G} g = c$ for all h. Moreover, with $e = c/|G|$, we have $e^2 = e$. Thus, $\mathbb{C}[G]e$ is a one-dimensional representation where each element of G acts as the identity; that is, it's the trivial representation.

It may be very difficult to construct idempotents that generate all irreducible representations (unless one knows these representations in advance). However, for the group S_n there is an explicit construction that we will now describe. We stress that we will not actually *use* Proposition 1; it is only included here for motivation.

The starting point for our construction is a partition λ of n, that is, a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ of positive integers with $\sum_j \lambda_j = n$. Partitions are conveniently pictured as *Young diagrams.* To explain this, the following example should be sufficient.

$\lambda = (4, 2, 1)$



By a *tableau* (pl. tableaux), we mean a filling of the boxes of a Young diagram with the symbols $1, \dots, n$, such that each symbol is used exactly once. The *canonical tableau* is the particular tableau defined as in the following example.

1	2	3	4
5	6		
7			

Let P denote the subgroup of S_n consisting of permutations fixing the symbols in each row of the canonical tableau and Q the subgroup fixing the symbols in each column. In the above example, $P \simeq S_4 \times S_2$ consists of permutations of $\{1, \dots, 7\}$ fixing $\{1, 2, 3, 4\}$, $\{5, 6\}$ and $\{7\}$ and $Q \simeq S_3 \times S_2$ is the permutations fixing $\{1, 5, 7\}$, $\{2, 6\}$, $\{3\}$ and $\{4\}$. Now let

$$c_\lambda = \sum_{p \in P} p \sum_{q \in Q} \operatorname{sgn}(q) q \in \mathbf{A}_n. \tag{2}$$

This is known as a *Young symmetrizer.* Our main result is then as follows.

Theorem 2. *The irreducible representations of S_n are exactly the left ideals $V_\lambda = \mathbf{A}_n c_\lambda$, where λ ranges over all partitions of n.*

The one-row partition $\lambda = (n)$ corresponds to $c = \sum_{g \in S_n} g$. As we saw above, this gives the trivial representation. By a similar computation, the one-column partition $\lambda = (1, \dots, 1)$ corresponds to the alternating representation, which is one-dimensional and where each g acts by multiplication with $\mathrm{sgn}(g)$.

To give a less trivial example, the partition $\lambda = (n-1, 1)$ gives the *standard representation*, equivalent to the action by S_n on the space $V_n = \{x \in \mathbb{C}^n;\, x_1 + \dots + x_n = 0\}$. Let us check this explicitly when $n = 3$. In this case we have $P = \{\mathrm{id}, (12)\}$, $Q = \{\mathrm{id}, (13)\}$, so

$$c = (\mathrm{id} + (12))(\mathrm{id} - (13)) = \mathrm{id} + (12) - (13) - (132).$$

We compute

$$(13)c = (13) + (123) - \mathrm{id} - (23)$$

and let $V = \mathrm{span}\{c, (13)c\}$. We claim that $V_\lambda = V$; in particular, $\dim(V_\lambda) = 2$. Since S_3 is generated by (12) and (13) and obviously $(13)V = V$, it's enough to check that $(12)V \subseteq V$. To this end, we compute

$$(12)c = (12) + \mathrm{id} - (132) - (13) = c,$$

$$(12)(13)c = (132) + (23) - (12) - (123) = -c - (13)c.$$

We leave for the reader to check that $\pi : V_3 \to V_\lambda$ is intertwining, where $\pi(x_1, x_2, x_3) = x_1(13)c + x_2(23)c + x_3 c$.

Having constructed the irreducible representations of S_n, we want to compute the character table. A conjugacy class of S_n consists of all elements containing exactly c_k k-cycles for each k, where c_k are non-negative integers subject to

$$c_1 + 2c_2 + \dots + nc_n = n.$$

Equivalently, a conjugacy class is described by the partition containing k copies of the number c_k for each k. (The partition pictured above corresponds to permutations in S_7 composed of a 4-cycle, a 2-cycle and a trivial cycle.) This gives a natural correspondence between irreducible representations and conjugacy classes, as they are both labelled by partitions. No such correspondence is known for general groups.

The following result describes the character table of S_n.

Theorem 3. *Let χ_λ be the irreducible charcacter of S_n labelled by the partition λ and C the conjugacy class labelled by the numbers c_k. Let $l_j = \lambda_j + m - j$. Then, $\chi_\lambda(C)$ is the coefficient of $x_1^{l_1} x_2^{l_2} \cdots x_m^{l_m}$ in the polynomial*

$$\prod_{1 \le i < j \le m} (x_i - x_j) \prod_{j=1}^{n} (x_1^j + \dots + x_m^j)^{c_j}.$$

As an exampe, let us compute $\chi_\lambda(C)$, when $\lambda = (3,1)$ is the standard representation of S_2 and C is the conjugacy class containing $(12)(34)$. We then have $(l_1, l_2) = (4,1)$ and $c_j = 2\delta_{j,2}$. Thus, we should consider the polynomial

$$(x_1 - x_2)(x_1^2 + x_2^2)^2.$$

Identifying the coefficient of $x_1^4 x_2$, we find that $\chi_\lambda(C) = -1$.

As a special case of Theorem 3, $\dim(V_\lambda) = \chi_\lambda(\mathrm{id})$ is the coefficient of $x_1^{l_1} \cdots x_m^{l_m}$ in

$$(x_1 + \cdots + x_m)^n \prod_{1 \le i < j \le m} (x_i - x_j).$$

It is not hard to write down these coefficients explicitly (we leave the details as an exercise). Namely,

$$\dim V_\lambda = \frac{n!}{l_1! \cdots l_m!} \prod_{1 \le i < j \le m} (l_i - l_j),$$

where l_j are as in Theorem 3. There is an attractive way to present this identity in terms of *hook lengths* (again we leave the details to the reader). The hook length of a box in a Young diagram is the number of boxes in the "hook" consisting of the box itself and all boxes that are straight below it or straight to the right of it.

Corollary 4. *If V_λ is the irreducible representation of S_n corresponding to the partition λ, then $\dim V_\lambda = n!/H$, where H is the product of the hook lengths of all the boxes in the Young diagram for λ.*

As an example, in the following diagram, each box is labelled by its hook length. The corresponding representation has dimension $7!/6 \cdot 4 \cdot 3 \cdot 2 = 35$.

$$\begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Irreducible representations

In this section, we prove Theorem 2. We first give an intrinsic description of the set PQ.

Lemma 5. *A permutation $g \in PQ$ if and only if there is no pair of labels (i, j) in the same column in the canonical tableau such that $(g(i), g(j))$ are in the same row.*

Proof. Suppose $g = pq$ with $p \in P$ and $q \in Q$ and let $i \neq j$ be in the same column. Then, $q(i) \neq q(j)$ are still in the same column, hence in distinct rows. It follows that $g(i) = p(q(i))$ and $g(j) = p(q(j))$ are in distinct rows.

For the converse, suppose g has the mentioned property. Let $g(T)$ be the tableau obtained from the canonical tableau by replacing each label i by $g(i)$. Then, there is no pair of labels that are in the same row of T and in the same column of $g(T)$. This means that we can reorder the elements within each row of T such that they end up in the same column as in $g(T)$, and then obtain $g(T)$ by reordering the elements within each column. In terms of permutations, we go from T to $g(T)$ by first applying a permutation $p \in P$ and then a permutation pqp^{-1} for some $q \in Q$. Then, $g = pq$. □

We introduce the notation

$$a_\lambda = \sum_{g \in P} g, \qquad b_\lambda = \sum_{q \in Q} \operatorname{sgn}(q) q,$$

so that $c_\lambda = a_\lambda b_\lambda$. The following result is the key for showing that V_λ is irreducible.

Lemma 6. *For any $x \in \mathbf{A}_n$, $a_\lambda x b_\lambda \in \mathbb{C} c_\lambda$.*

Proof. By linearity, we may assume $x \in S_n$. Suppose first that $x = pq \in PQ$. Since $a_\lambda p = a_\lambda$ and $q b_\lambda = \operatorname{sgn}(q) b_\lambda$, we find that $a_\lambda x b_\lambda = \operatorname{sgn}(q) c_\lambda$. To complete the proof, we show that if $x \notin PQ$, then $a_\lambda x b_\lambda = 0$. By Lemma 5, we can find a pair (i, j) in the same column of the standard tableau such that $(x(i), x(j))$ are in the same row. Let $p = (x(i)\, x(j)) \in P$. Then, $q = x^{-1} p x = (ij) \in Q$. This gives

$$a_\lambda x b_\lambda = a_\lambda p x b_\lambda = a_\lambda x q b_\lambda = \operatorname{sgn}(q) a_\lambda x b_\lambda = -a_\lambda x b_\lambda,$$

so $a_\lambda x b_\lambda = 0$. □

Choosing $x = b_\lambda a_\lambda$ gives the following important fact.

Corollary 7. *The element c_λ is proportional to an idempotent.*

We define the lexicographic ordering of partitions by $\lambda > \mu$ if there exists j such that $\lambda_i = \mu_i$ for $1 \leq i < j$ and $\lambda_j > \mu_j$. This is a total ordering; that is, if $\lambda \neq \mu$ then either $\lambda > \mu$ or $\mu > \lambda$. We will need the following simple combinatorial fact.

Lemma 8. *Let T_λ and T_μ be two tableaux (not necessarily canonical) on two Young diagrams λ and μ with $\lambda > \mu$. Then, we can find a pair (i, j) such that i and j appear in the same row of T_λ and the same column of T_μ.*

Proof. Suppose for a contradiction that there is no such pair (i, j). Then, all the λ_1 symbols in the first row of T_λ must be in different columns in T_μ. Since there are μ_1 such columns, we have $\lambda_1 \leq \mu_1$ and consequently $\lambda_1 = \mu_1$. We may then reorder the symbols within each column of T_μ so that the first row of T_λ and T_μ agree. This does not affect our assumption that there is no pair (i, j) with the stated property. Repeating the same argument for each row we eventually find that $\lambda = \mu$. $\square$

The following result is the key for showing that the representations V_λ are mutually non-equivalent.

Lemma 9. *If $\lambda > \mu$ then $a_\lambda \mathbf{A}_n b_\mu = 0$.*

Proof. This can be proved in the same way as the case $g \notin PQ$ in Lemma 6. It suffices to show that $a_\lambda g b_\mu = 0$ for $g \in S_n$. We choose T_λ as the canonical tableau on λ and T_μ as the tableau obtained from the canonical tableau on μ by replacing each i by $g(i)$. Picking a pair (i, j) as in Lemma 8, we have $p = (ij) \in P_\lambda$, $q = g^{-1}pg = (g^{-1}(i)g^{-1}(j)) \in Q_\mu$, where we indicate the dependence of the groups P and Q on the underlying partition. Then,

$$a_\lambda g b_\mu = a_\lambda p q b_\mu = a_\lambda g q b_\mu = -a_\lambda g b_\mu,$$

so $a_\lambda b g_\mu = 0$. $\square$

We are finally ready for our main result.

Proof of Theorem 2. We first prove that $V_\lambda \neq \{0\}$. To this end, note that any element in PQ can be written uniquely as pq, with $p \in P$ and $q \in Q$. Indeed, if $p_1 q_1 = p_2 q_2$, then $p_2^{-1} p_1 = q_2 q_1^{-1} \in P \cap Q = \{\mathrm{id}\}$. Thus, the terms in (2) are linearly independent, so $0 \neq c_\lambda \in V_\lambda$.

By Corollary 7, $V_\lambda = \mathbf{A}_n e_\lambda$ for some idempotent e_λ proportional to c_λ. Let $\phi \in \mathrm{Hom}_{S_n}(V_\lambda, V_\mu)$. Then, $\phi(e_\lambda) = xe_\mu$ for some $x \in \mathbf{A}_n$. It follows that

$$\phi(ye_\lambda) = \phi(ye_\lambda^2) = ye_\lambda\phi(e_\lambda) = ye_\lambda xe_\mu, \qquad y \in \mathbf{A}_n. \tag{3}$$

If $\lambda = \mu$, it follows from Lemma 6 that $e_\lambda xe_\mu$ is proportional to e_λ. Thus, $\mathrm{End}_{S_n}(V_\lambda) = \mathbb{C}\,\mathrm{Id}$. It follows that V_λ is irreducible (if we can split a representation as $V = V_1 \oplus V_2$, then the projections to V_j are linearly independent intertwining maps). Here we are using that $V_\lambda \neq 0$.

If $\lambda > \mu$ it follows from Lemma 9 that $e_\lambda xe_\mu = 0$. Thus, $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = 0$, which implies $V_\lambda \not\simeq V_\mu$. We have now found as many non-equivalent irreducible representations as there are conjugacy classes in S_n, so there can be no others.

□

Characters

In this section, we prove Theorem 3. To compute the character of $V_\lambda = \mathbf{A}_n a_\lambda b_\lambda$, we will first consider the representation $U_\lambda = \mathbf{A}_n a_\lambda$, which is much easier to understand. We first note that the argument used in the proof of Theorem 2 gives the following result.

Lemma 10. *Let $U_\lambda = \mathbb{C}[G]a_\lambda$. Then, the decomposition of U_λ into irreducibles has the form $U_\lambda = V_\lambda \oplus \bigoplus_{\mu>\lambda} K_{\mu\lambda} V_\mu$.*

The numbers $K_{\mu\lambda}$ are called *Kostka numbers*; they can be defined combinatorially as the number of a certain kind of generalized tableaux.

Proof. It suffices to prove that $\dim \mathrm{Hom}_{S_n}(U_\lambda, V_\mu)$ is 1 for $\mu = \lambda$ and 0 for $\mu < \lambda$. We first note that $U_\lambda = \mathbf{A}_n e_\lambda$, where $e_\lambda = a_\lambda/|P|$ is an idempotent. If $\phi \in \mathrm{Hom}_{S_n}(U_\lambda, V_\mu)$, then $\phi(e_\lambda) = xc_\lambda$ for some $x \in \mathbf{A}_n$. Just as in (3), it follows that $\phi(ye_\lambda) = ye_\lambda xc_\lambda$ for $y \in \mathbf{A}$. The conclusion now follows from Lemma 6 and Lemma 9. □

We will now compute the character of U_λ. We find it instructive to first work in a more general setting, with $P \subseteq S_n$ replaced by arbitrary groups $H \subseteq G$. We define $\mathbb{C}[G/H]$ as the vector space with basis elements being the cosets gH, $g \in G$ (we do not assume that H is normal). Then, there is a natural representation π of G on $\mathbb{C}[G/H]$ defined by $\pi(g_1)g_2H = g_1g_2H$. (The technical term for π is the representation of G induced from the trivial representation of H.)

Lemma 11. *If* $a = \sum_{h \in H} h \in \mathbb{C}[G]$*, then* $\mathbb{C}[G]a \simeq \mathbb{C}[G/H]$ *as representations of* G*.*

Proof. Note that $ha = a$ for $h \in H$. Thus, $gH \mapsto ga$ is a well-defined map $G/H \to \mathbb{C}[G]a$. We extend it linearly to $\Phi : \mathbb{C}[G/H] \to \mathbb{C}[G]a$. Then, Φ is clearly surjective and intertwining, so it remains to show that it's injective. To this end, let $g_j \in G$ be representatives for the cosets, so that any $x \in \mathbb{C}[G/H]$ can be written $x = \sum_j c_j g_j H$, $c_j \in \mathbb{C}$. Then $\Phi(x) = \sum_j \sum_{h \in H} c_j g_j h$. Since the elements $g_j h$ in this sum are mutually distinct, it follows that $\mathrm{Ker}(\Phi) = 0$. □

By the following lemma, computing the character of $\mathbb{C}[G/H]$ can be reduced to a counting problem.

Lemma 12. *The character* χ *of* $\mathbb{C}[G/H]$ *is given by*

$$\chi(C) = \frac{|G|\,|C \cap H|}{|H|\,|C|},$$

where C *is any conjugacy class of* G*.*

Proof. Since G acts by permuting the basis elements gH, the character counts the number of basis elements fixed by the group action. Thus, for $x \in C$,

$$\begin{aligned}\chi(x) &= |\{gH \in G/H;\, xgH = gH\}| = |\{gH \in G/H;\, g^{-1}xg \in H\}| \\ &= \frac{1}{|H|}\,|\{g \in G;\, g^{-1}xg \in H\}|.\end{aligned}$$

Consider for fixed y the equation $g^{-1}xg = y$. Clearly, it has a solution g only if $y \in C$. Apart from that restriction, the number of solutions does not depend on y. This is because if $y \in C$ then $y = \tilde{g}^{-1}x\tilde{g}$ for some $\tilde{g} \in G$. It follows that $g^{-1}xg = y$ if and only if $(\tilde{g}g^{-1})^{-1}x(\tilde{g}g^{-1}) = x$. Thus, for any $y \in C$, the equation has $|G|/|C|$ solutions. Consequently,

$$|\{g \in G;\, g^{-1}xg \in H\}| = \frac{|H \cap C||G|}{|C|}.$$

□

It follows from the proof that if $x \in C$ and $G_x = \{g \in G;\, gxg^{-1} = x\}$, then

$$|G| = |G_x|\,|C|. \tag{4}$$

This is a special case of the orbit-stabilizer theorem for general group actions. We will now use it to compute the number of elements in each conjugacy class of S_n. Recall that such a class consists of all elements containing exactly c_k k-cycles for each k, where c_k are non-negative integers subject to

$$c_1 + 2c_2 + \cdots + nc_n = n.$$

Recall also that conjugation of permutations corresponds to permuting the names of the symbols that the permutations act on. For instance, if

$$x = (123)(45)(678)$$

and $g = (135)$, then

$$\begin{aligned} gxg^{-1} &= (g(1)g(2)g(3))(g(4)g(5))(g(6)g(7)g(8)) = (325)(41)(678) \\ &= (14)(253)(678). \end{aligned}$$

Suppose that $gxg^{-1} = x$ for some $x \in C$. Then, G must move the symbols occuring in each cycle to the symbols in another (or the same) cycle of the same length. The number of such permutations of the cycles is $c_1!c_2! \cdots c_n!$. Having fixed how the cycles are permuted, we may cyclically reorder the symbols within each cycle without changing x. For each k-cycle there are k ways of doing this, so the total number of reorderings is $1^{c_1}2^{c_2} \cdots n^{c_n}$. Thus, we obtain from (4) the enumeration

$$|C| = \frac{n!}{\prod_{k=1}^{n} k^{c_k} c_k!}. \tag{5}$$

To give an example, if $x = (123)(45)(678)$ as above and $gxg^{-1} = x$, then g must fix the 2-cycle (45) and either fix or interchange the 3-cycles (123) and (678). For instance, we can interchange them using $g = (16)(27)(38)$. Another g that will work is $g = (132)$, since it maps the cycle (123) to $(312) = (123)$ and fixes the other cycles.

Let us now consider the subgroup $H = S_{\lambda_1} \times \cdots \times S_{\lambda_m}$ of $G = S_n$, where $\sum_j \lambda_j = n$. (If $\lambda_1 \geq \cdots \geq \lambda_m$, this is the subgroup previously denoted P.) The conjugacy classes in H are direct products of conjugacy classes in S_{λ_i}. Suppose that the i-th component contains c_{ij} cycles of length j. Then,

$$\begin{cases} c_{11} + 2c_{12} + \cdots + nc_{1n} = \lambda_1, \\ \quad \vdots \\ c_{m1} + 2c_{m2} + \cdots + nc_{mn} = \lambda_m. \end{cases} \tag{6a}$$

Using (5) for each component, we find that this conjugacy class has size

$$\prod_{j=1}^{m} \frac{\lambda_j!}{\prod_{k=1}^{n} k^{c_{jk}} c_{jk}!}.$$

Let us now take a conjugacy class C in G. Then, $C \cap H$ splits as a union of conjugacy classes in H. Explicitly, if the elements in C contain exactly c_k cycles of length k, then $C \cap H$ is a union of conjugacy classes corresponding to (6a), where

$$\begin{cases} c_{11} + c_{21} + \cdots + c_{m1} = c_1, \\ \quad \vdots \\ c_{1n} + c_{2n} + \cdots + c_{mn} = c_n. \end{cases} \tag{6b}$$

It follows that

$$|C \cap H| = \sum_{(c)} \prod_{j=1}^{m} \frac{\lambda_j!}{\prod_{k=1}^{n} k^{c_{jk}} c_{jk}!} = \frac{\prod_{j=1}^{m} \lambda_j!}{\prod_{k=1}^{n} k^{c_k}} \sum_{(c)} \frac{1}{\prod_{j=1}^{m} \prod_{k=1}^{n} c_{jk}!}, \tag{7}$$

where the sum is over all non-negative integer solutions to (6). We can now compute $\chi(C)$ from Lemma 12. Using $|G| = n!$, $|H| = \prod_{j=1}^{m} \lambda_j!$, (5) and (7), we find after simplification that

$$\chi(C) = \sum_{(c)} \prod_{k=1}^{n} \frac{c_k!}{\prod_{j=1}^{m} c_{jk}!}.$$

By the multinomial theorem, this is equivalent to the following result.

Lemma 13. *We have the generating function*

$$\prod_{k=1}^{n} (x_1^j + \cdots + x_m^j)^{c_j} = \sum_{\lambda_1 + \cdots + \lambda_m = n} \psi_\lambda(C) x_1^{\lambda_1} \cdots x_m^{\lambda_m}$$

for the characters ψ_λ of $\mathbb{C}[G/H]$, with $G = S_n$, $H = S_{\lambda_1} \times \cdots \times S_{\lambda_m}$ (if some $\lambda_j = 0$, the factor S_{λ_j} is deleted) and C the conjugacy class consisting of permutations with exactly c_k k-cycles.

If λ is a partition, that is, $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$, then ψ_λ is the character of $U_\lambda = \mathbf{A}_n a_\lambda$. In general, since the modules $\mathbb{C}[G/H]$ are clearly isomorphic under permutation of the indices λ_j, ψ_λ is the character of $\mathbf{A}_n a_{[\lambda]}$, where $[\lambda]$

is the partition obtained by reordering λ and ignoring zeroes (for instance, if $\lambda = (3, 0, 2, 0, 3)$, then $[\lambda] = (3, 3, 2)$).

Let ϕ_λ denote the expression for χ_λ proposed in Theorem 3 (we want to prove that $\chi_\lambda = \phi_\lambda$). Comparing with Lemma 13, we can relate ϕ_λ to ψ_λ if we can expand the product $\prod_{1\le i<j\le m}(x_i - x_j)$ into monomials. That is a well-known result.

Lemma 14 (Vandermonde determinant)**.** *We have*

$$\prod_{1\le i<j\le m} (x_i - x_j) = \det_{1\le i,j\le m} (x_i^{m-j}). \tag{8}$$

This can be proved by a variety of methods. In the context of the present course, a very natural approach is as follows. Call a polynomial in several variables *anti-symmetric* if it changes sign whenever two variables are interchanged. If V_m denotes the polynomials in one variable of degree $\le m-1$, then it's easy to see that $\bigwedge^k V_m$ is isomorphic to the space of anti-symmetric polynomials in k variables that are of degree $\le m-1$ in each variable. Obviously, both sides of (8) belong to the space $\bigwedge^m V_m$. But since $\dim V_m = m$, that space is one-dimensional, so (8) holds up to a multiplicative constant. To identify that constant, check that the coefficient of $x_1^{m-1}x_2^{m-2}\cdots x_{m-1}$ in both sides of (8) is 1.

Let us write $\rho = (m, m-1, \dots, 0)$. In obvious multi-index notation, we then have

$$\prod_{1\le i<j\le m} (x_i - x_j) \prod_j (x_1^j + \cdots + x_m^j)^{c_j} = \sum_\lambda \theta_\lambda(C) x^{\lambda+\rho} + R, \tag{9}$$

where the remainder R collects terms that are not of the form $\lambda+\rho$ with λ a partition. By Lemma 13 and Lemma 14, the left-hand side of (9) can be written

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) x^{\sigma(\rho)} \sum_\lambda \psi_\lambda(C) x^\lambda.$$

It follows that

$$\theta_\lambda = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\psi_{\lambda+\rho-\sigma(\rho)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\psi_{[\lambda+\rho-\sigma(\rho)]}. \tag{10}$$

Again, we will need a simple combinatorial fact.

Lemma 15. *If λ is a partition and σ a permutation then, in the notation explained above, $[\lambda + \rho - \sigma(\rho)] \geq \lambda$, with equality if and only if $\sigma = \mathrm{id}$.*

Proof. Let P_+ denote all positive linear combination of the vectors $e_i - e_j$, with $i < j$. We claim that $[\lambda + \rho - \sigma(\rho)] \in \lambda + P_+$, which clearly implies $[\lambda + \rho - \sigma(\rho)] \geq \lambda$. To see this, first write

$$\begin{aligned}\rho - \sigma(\rho) = (\rho_1 - \sigma(\rho)_1)(e_1 - e_2) + (\rho_1 + \rho_2 - \sigma(\rho)_1 - \sigma(\rho)_2)(e_2 - e_3) \\ + \cdots + (\rho_1 + \cdots + \rho_m - \sigma(\rho)_1 - \cdots - \sigma(\rho)_m)e_m.\end{aligned}$$

Since $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$, the coefficients in this expansion are non-negative, and the coefficient of e_m is 0. Thus, $\lambda + \rho - \sigma(\rho) \in \lambda + P^+$. Next, if $\mu \in \mathbb{Z}^m_{\geq 0}$ is arbitrary, then $[\mu]$ is obtained from μ by successively interchanging pairs (μ_i, μ_j), where $i < j$ and $\mu_i < \mu_j$. This is achieved by adding the vector $(\mu_j - \mu_i)(e_i - e_j) \in P^+$. Thus, we always have $[\mu] \in \mu + P^+$. This completes the proof, as it also follows from the above discussion that we have equality only if $\rho = \sigma(\rho)$, that is, $\sigma = \mathrm{id}$. $\square$

Consequently, we can write (10) as

$$\theta_\lambda = \psi_\lambda + \sum_{\mu > \lambda} L_{\mu\lambda} \psi_\mu,$$

for some integers $L_{\lambda\mu}$. Moreover, by Lemma 10,

$$\psi_\lambda = \chi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu,$$

so

$$\theta_\lambda = \chi_\lambda + \sum_{\mu > \lambda} M_{\mu\lambda} \chi_\mu.$$

Since the characters are orthonormal in $L^2(S_n)$,

$$\langle \theta_\lambda, \theta_\lambda \rangle = 1 + \sum_{\mu > \lambda} M_{\mu\lambda}^2.$$

Thus, to complete the proof that $\theta_\lambda = \chi_\lambda$, we only have to prove that $\langle \theta_\lambda, \theta_\lambda \rangle = 1$. This is an entertaining computation.

By (5),

$$\langle \theta_\lambda, \theta_\lambda \rangle = \frac{1}{n!} \sum_C |C| \, \theta_\lambda(C)^2 = \sum_{c_1 + 2c_2 + \cdots + nc_n = n} \frac{\theta_\lambda(C)^2}{\prod_{k=1}^n k^{c_k} c_k!}.$$

In view of the definition of θ_λ, we introduce the generating function

$$S(x,y) = \sum_{c_1,c_2,c_3,\dots} \prod_{k=1}^{\infty} \frac{(x_1^k + \cdots + x_m^k)^{c_k}(y_1^k + \cdots + y_m^k)^{c_k}}{k^{c_k} c_k!}.$$

(We consider this as a formal power series; in particular, convergence aspects are irrelevant.) Then, $\langle \theta_\lambda, \theta_\lambda \rangle$ is the coefficient of $x^{\lambda+\rho} y^{\lambda+\rho}$ in

$$\prod_{1 \le i < j \le m} (x_i - x_j)(y_i - y_j) S(x,y).$$

Recall that if we write $\exp(x) = \sum_{k=0}^{\infty} x^k/k!$ and $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$, then $\exp(x+y) = \exp(x)\exp(y)$ and $\exp(-\log(1-x)) = 1/(1-x)$ in the ring of formal power series. It follows that

$$\begin{aligned} S(x,y) &= \prod_{k=1}^{\infty} \exp\left(\frac{(x_1^k + \cdots + x_m^k)(y_1^k + \cdots + y_m^k)}{k} \right) \\ &= \prod_{i,j=1}^{m} \exp\left(\sum_{k=1}^{\infty} \frac{x_i^k y_j^k}{k} \right) = \prod_{i,j=1}^{m} \frac{1}{1 - x_i y_j}. \end{aligned}$$

We know apply the Cauchy determinant evaluation. (Replace y_j by $-y_j^{-1}$ to get the form that appeared in a previous exercise.)

Lemma 16 (Cauchy determinant)**.** *We have*

$$\det_{1 \le j,k \le m} \left(\frac{1}{1 - x_j y_k} \right) = \frac{\prod_{1 \le j < k \le m} (x_j - x_k)(y_j - y_k)}{\prod_{j,k=1}^{m} (1 - x_j y_k)}.$$

One can obtain a short proof similarly to Lemma 14; simply note that if we multiply by the denominator both sides are polynomials in the same one-dimensional space; the multiplicative constant is most easily computed inductively by looking at the limit of both sides when $x_1 y_1 \to 1$.

We now know that $\langle \theta_\lambda, \theta_\lambda \rangle$ is the coefficient of $x^{\lambda+\rho} y^{\lambda+\rho}$ in the formal power series expansion of

$$\begin{aligned} \det_{1 \le j,k \le m} \left(\frac{1}{1 - x_j y_k} \right) &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{j=1}^{m} \frac{1}{1 - x_j y_{\sigma(j)}} \\ &= \sum_{\sigma \in S_m} \sum_{k_1,\dots,k_m \ge 0} \operatorname{sgn}(\sigma) x_1^{k_1} \cdots x_m^{k_m} y_{\sigma(1)}^{k_1} \cdots y_{\sigma(m)}^{k_m}. \end{aligned}$$

It is important to note that the indices $(\lambda + \rho)_i$ are mutually distinct. Thus, only the permutation σ contributes to the coefficient of $x^{\lambda+\rho}y^{\lambda+\rho}$, and we find that this coefficient is 1. This completes the proof of Theorem 3.