Multilinear algebra 2015. Check-list for theory.

This is a list of the main definitions and theorems that are included in the course. As for the proofs, you should use your own judgement and study the proofs that seem most important/interesting/instructive. On the exam you may be asked to write down the proof of a result that we learned, but such questions will not form a big part of the exam.

Fundamental notions of group and ring theory (prerequisites)

- Basic notions of group theory: Group, abelian/commutative group, group homomorphism, subgroup, coset, normal subgroup, quotient group, direct product of groups.
- Examples of groups: Cyclic groups ($\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$), symmetric group, general linear group.
- Fundamental theorem of group homomorphisms: $G/\operatorname{Ker}(\phi) \simeq \operatorname{Im}(\phi)$.
- Lagrange's theorem: The order of a subgroup divides the order of the group.
- Cayley's theorem: Every finite group can be embedded into S_n.
- What is the cycle decomposition of a permutation? How does one compute with cycle decompositions (group multiplication, conjugation). What are the conjugacy classes of the group S_n?
- Basic notions of ring theory: Ring, ring homomorphism, subring, ideal, left ideal, quotient, direct product. (There are several competing definitions of ring, on the exam a ring is an associative ring with unit unless explicitly stated otherwise.)
- Examples of rings: Number fields, residue class rings, polynomials, matrices.
- Special types of rings: Commutative ring, Integral domain, UFD, PID, Field. To which types do e.g. $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{Z}[x, y]$, $\mathbb{R}$, $\mathbb{R}[x]$, $\mathbb{R}[x, y]$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/7\mathbb{Z}$ belong?
- Types of ideals: Finitely generated, principal, maximal, prime.
- Fundamental theorem of ring homomorphisms.
- What can you say about R/I when I is prime and when I is maximal?

More advanced notions of group and ring theory

- Group ring and group algebra of a group.
- Irreducible element and prime element in an integral domain.
- Relation between unique factorization and irreducible elements being prime.

- Every PID is Noetherian.
- Every PID is an UFD.

Modules

- Basic notions of module theory: Module, Module homomorphism, Submodule, Quotient module, Direct sum/product, Dual module, Free module, Basis for a module, Torsion module, Torsion-free module.
- Why is a $\mathbb{Z}$-module the same thing as an abelian group?
- Fundamental theorem of module homomorphisms.
- If two free modules have bases with the same cardinality, are they then isomorphic? If two free modules are isomorphic, do any two bases for them necessarily have the same cardinality? (Answer: Yes, No.)
- Prove that for a finitely generated free module over a commutative ring, any two bases have the same cardinality. (See lecture notes for a pedestrian proof and Brzezinski for a more abstract proof that does not assume finitely generated.)
- Characterization of direct sum of two modules (Lemma 2 in lecture notes).
- Exact sequence, short exact sequence, split exact sequence.
- Relation between split exact sequences and direct sums.
- Prove that if M/L is free, then L is a direct summand of M.

Modules over a PID

- Classification of finitely generated modules over a PID; classification of finitely generated abelian groups; classification of finite abelian groups.
- What do the numbers $\dim_{R/pR}(p^k M/p^{k+1}M)$ (with $p \in R$ a prime element) say about M?
- What is the primary decomposition and the invariant factor decomposition? How to pass between them?
- How can we use the general theory of modules over a PID to study matrices?
- What is the primary canonical form and rational canonical form of a matrix? How can they be obtained in practice? Do they change if we change the base field?
- What is the Jordan canonical form of a matrix (over $\mathbb{C}$)? How can it be obtained in practice?

- What is the minimal polynomial of a matrix? How can we find the minimal polynomial and characteristic polynomial given the rational canonical form or the Jordan canonical form?

- Cayley–Hamilton theorem.

Tensor products

- Define the tensor products of modules over a commutative ring. Give both a construction and a characterization by a universal property.

- For vector spaces with given bases, write down a basis for the tensor product.

- Basic properties of tensor products: Commutativity, associativity, distributivity with respect to direct sums.

- Define the tensor product of module homomorphisms. How does it look in a basis, that is, what is the tensor product of two matrices?

- How can tensor products be used to "extend scalars"?

- For finite-dimensional vector spaces, $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ and $\mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2) \simeq \mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2)$.

- What is a tensor on a finite-dimensional vector space (Answer: an element of $\bigotimes^k V \otimes \bigotimes^l V^*$)? Explain how linear and multilinear maps can be identified with tensors. How can tensors be represented by multi-dimensional matrices, and how do these matrices transform under a change of coordinates?

- What is an alternating multilinear map? What's special about characteristic 2?

- What is the exterior tensor product? Give both a construction and a characterization by a universal property.

- If $u \in \bigwedge^k V$ and $v \in \bigwedge^l V$, how do we define $u \wedge v \in \bigwedge^{k+l} V$? With this definition, do we always have $u \wedge v = -v \wedge u$? (Answer: No!)

- What is the exterior power $\bigwedge^k f$ of a linear map?

- Given a basis for a vector space V, write down a basis for $\bigwedge^k V$. Given a linear endomorphism f of V with matrix A, what is the matrix $\bigwedge^k A$ representing the map $\bigwedge^k f$?

Group representations

- Basic notions: Group representation, Intertwining map (homomorphism of group representations), Isomorphism of group representations, Subrepresentation, Irreducible representation, Class function, Character, Trivial representation, Regular representation.

- Relation between representations of a group and representations (=modules) of the group algebra.
- If V and W are representations of G, how do we define a representation of G on V^*, $V \oplus W$, $V \otimes W$ and $\mathrm{Hom}(V, W)$? What are the characters of all those representations?
- Maschke's theorem. Does it hold over finite fields? (Answer: No, we need $\mathrm{char}(K) \nmid |G|$.)
- Schur's lemma, especially over $\mathbb{C}$.
- How does the regular representation split into irreps?
- How is the order of the group related to the dimension of the irreps?
- Why does a finite group have as many conjugacy classes as irreps?
- Why is a representation determined by its character?
- Orthogonality of characters (there are two such relations, which are dual to each other).
- Decomposition of the group algebra as a representation of $G \times G$ and as an associative algebra.
- Peter–Weyl theorem (for finite groups).
- Every irrep is a left ideal $\mathbb{C}[G]e$, with $e^2 = e$ (see lecture notes on symmetric group).
- Given the character table, how do we find: The order of the group, the order of the conjugacy classes, the normal subgroups, the center of the group, the commutator subgroup.
- Describe the representations of a finite abelian group.
- If H is a (not necessarily normal) subgroup of G, explain how to make $\mathbb{C}[G/H]$ a representation of G and compute its character (see notes on symmetric group).

Representations of the symmetric group

- How many elements are there in a conjugacy class of S_n?
- How can one construct all irreps of S_n?
- How can one compute the character table of S_n?
- What's the dimension of the irreps?
- How does $\bigotimes^n V$ decompose into irreps under the natural action of the groups S_n, $\mathrm{GL}(V)$ and $S_n \times \mathrm{GL}(V)$?
- What are the character of those irreps of $\mathrm{GL}(V)$ that appear in $\bigotimes^n V$?